On the Aggregation of Risk

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Abstract. The objective of this paper is to propose a general framework for aggregating economic capital across risk types. Our starting point are aggregation models that operate in a single-period framework, typically with a planning horizon of one year. As an example, we present Deutsche Bank's EC aggregation model including calibration techniques for correlation parameters. The second part of the paper focuses on the development of multi-period extensions of the traditional single-period approach. We argue that multi-period models provide the natural setting for aggregating risk types with different liquidity profiles. Several rollover and risk management strategies are presented and their impact is analyzed in a number of examples.

Key words: risk diversification, economic capital, factor model, copula, liquidity, multi-period Monte Carlo simulation

1 Introduction

A recent industry survey, co-sponsored by the International Financial Risk Institute and the Chief Risk Officers Forum, has confirmed that economic capital is becoming a core part of the financial steering and management of banks and insurance companies. In particular, economic capital is moving out of the risk function into broader management decisions and into discussions with key external stakeholders. The IFRI/CRO Forum survey (2007) also shows a general convergence of the economic capital framework although non-financial risks, i.e., operational and business risk, are lagging behind.

The survey identified the treatment of diversification effects between risk types as a major area for further discussion within the industry. Currently, the quantitative methodologies as well as the correlation estimates vary widely across institutions with little sign of a single approach holding sway. It is therefore not surprising that estimates of the impact of diversification on aggregate economic capital differ significantly: between 10% - 30% in the banks and 21% - 59% in the insurance companies participating in the survey.

We will now review the main approaches to the aggregation of economic capital across risk types that have been proposed in the literature. In many financial institutions, the calculation of total economic capital is performed in two steps. First, economic capital is calculated for individual risk types, most prominently for credit, market and operational risk. In a second step, the stand-alone EC figures are aggregated to obtain the total economic capital of the bank. The most straightforward aggregation technique is summation of stand-alone figures, e.g.

$$\mathrm{EC(total)} = \mathrm{EC(credit)} + \mathrm{EC(market)} + \mathrm{EC(operational)}.$$

This approach allocates capital for all risk types separately covering the case that worst case scenarios happen simultaneously. Since imperfect correlations of risk types are not taken into account the

[1]The views expressed in this paper are those of the authors and do not necessarily reflect the position of Deutsche Bank AG.

simple summation of standalone EC has always been considered as overly conservative, i.e., a bank faces less risk and would therefore require less capital to operate safely than suggested by the sum of the individual risks. This view has been recently challenged, for example in Breuer et al. (2007) and Basel Committee on Banking Supervision (2009). In these articles it is pointed out that non-linear interactions between market and credit risk may lead to compounding effects, which are not captured in standalone risk assessments for credit and market risk. An example is the complex interaction between market and credit risk in structured products, which contributed to significant writedowns during the current financial crisis. On the other hand, it is well documented that in many portfolios large diversification effects exist between credit and market risk, e.g. Rosenberg and Schuermann (2006) and Drehmann et al. (2008). Another important source of diversification is the imperfect correlation between operational risk and the other risk types. In order to quantify these diversification effects a formal dependence structure between the different risk types has been introduced in a number of papers. The main approaches in the literature are (see, for example, Saita (2004) for a comparison):

1. classic square root formula,
2. multi-factor approach (base-level aggregation),
3. aggregation through copulas (top level aggregation).

The square root formula is an analytic approach based on the covariance structure of the joint distribution of the different risks, see, for example, Kuritzkes et al. (2002). It only holds for elliptic distributions. Despite this limitation it is rather popular in practice (IFRI/CRO Forum survey, 2007), mainly due to its simplicity. We refer to Rosenberg and Schuermann (2006) and Böcker and Hillebrand (2007) for a comparison to other risk aggregation techniques.

In the multi-factor (or base-level) aggregation approach, the economic risk factors are identified that have most influence on the different risk types. A joint model is then developed for these risk factors, which includes a description of the dependence structure of the risk factors, for example through a copula. The losses related to the different risk types are specified by deterministic functions of the fluctuations in the risk factors. Hence, the marginal loss distributions are indirectly correlated through the relationship between the risk factors, see Alexander and Pezier (2003).

The starting point in the copula (or top-level) aggregation approach are the independently determined loss distributions for the individual risk types. In a second step, their dependence structure is defined via a copula function, i.e., the given loss distributions specify the marginals of the copula. The copula aggregation approach has been proposed in a number of papers, e.g. in Ward and Lee (2002), Dimakos and Aas (2004), Rosenberg and Schuermann (2006), Tang and Valdez (2006) and Böcker and Hillebrand (2007). Aas et al. (2007) use a multi-factor aggregation method for combining credit, market and ownership risk whereas operational and business risk are linked to the other risk types through a copula.

The objective of this paper is to present a general framework for aggregating economic capital across risk types. Our starting point is the aggregation model developed and implemented at Deutsche Bank, which is first presented in a multi-factor framework. We then show that it can also be interpreted as a copula model. In fact, it is an easy consequence of Sklar's theorem that each aggregate loss distribution specified in a multi-factor model can also be formalized through a copula. Hence, the purpose of classifying aggregate risk models in multi-factor models and copula models is not the definition of distinct mathematical model classes. It rather emphasizes a conceptual difference in the construction process of these models: copula models are built by aggregating given loss variables for different risk types, whereas multi-factor models start with dependent systematic risk factors and define loss functions for different risk types on top.

The risk aggregation model currently used in Deutsche Bank aggregates credit risk, operational risk and market risk, where market risk is split into three subtypes. A risk factor is defined for each risk type. The losses related to the different risk types are specified by deterministic functions of the fluctuations in the corresponding risk factors. It is assumed that risk factors follow a multi-dimensional normal distribution.

Particular emphasis is put on the specification of the correlation matrix of the risk factors since correlations or, more precisely, the dependencies between the different risk types are the key variables influencing the scale of diversification benefits. In Deutsche Bank's model, correlations between credit and market risk factors are derived from data series that are considered as reasonable proxies for the underlying economic relationships. The correlations between operational risk and the other risk types are derived from a qualitative analysis of high internal and external OR losses. The qualitative approach to the specification of OR correlations is motivated by the lack of historical OR data, compare to Aas et al. (2007) who also advocate OR model parameters based on expert opinions. The significance of OR correlations for risk measurement is highlighted in Rosenberg and Schuermann (2006).

Classical aggregation models operate in a single-period framework: separate loss distributions are specified for the different risk types at the planning horizon, typically one year, and then aggregated in order to obtain the diversified risk capital. However, this simple framework does not work well if some of the risk types have shorter planning horizons, for example due to the liquidity of the positions involved. In this paper, we propose a multi-period extension of the multi-factor aggregation model to improve the aggregation of risk types with different liquidity characteristics. The main idea is to estimate a liquidity horizon of l_j months for each risk type j and to represent its annual aggregate loss distribution as the sum of loss distributions covering $12/l_j$ intervals of l_j months. The liquidity horizon is chosen as the time period it takes to sell or hedge the risk inherent in the positions. The specification of the loss distributions for consecutive intervals is based on rollover assumptions, e.g. a constant level of risk or more sophisticated risk management strategies. Correlations between loss distributions of different risk types are implicitly specified by the dependence structure of the underlying risk factors.

The constant level of risk formalizes a rather simplistic risk management strategy: the portfolio is rebalanced at each rollover and the portfolio size is kept constant over the entire planning period regardless of the loss history. It seems more realistic, however, to assume that the portfolio is adjusted at the end of a rollover period based on the performance of the portfolio in the past, e.g. the portfolio size might be reduced after suffering high losses. The proposed multi-period aggregation model facilitates the implementation of sophisticated risk management strategies. We present strategies implemented on portfolio as well as asset level.

In a sequence of examples, the impact of different aggregation strategies is illustrated for a hypothetical bank. More precisely, we use a single-period model to compare the diversified EC of the bank under a Gaussian copula and a t-copula. A multi-period extension of the model serves as framework for rollover strategies based on a constant level of risk. It is shown that under standard modeling assumptions this rollover strategy significantly reduces the EC for credit risk (compared to a buy-and-hold strategy) whereas it has no impact on market risk EC. The final example presents an efficient EC reduction strategy for market risk, which scales down exposure if the aggregate loss exceeds a specified limit.

The paper has the following structure. In Section 2, we give a formal definition of multi-factor aggregation models in a single-period setting. Deutsche Bank's risk aggregation model is a specific instance of this model class. Its components are presented in Section 3: risk factors, their dependence structure and loss functions for the different risk types. Furthermore, it is shown that an equivalent model can be specified through a Gaussian copula. Section 4 deals with the extension of multi-factor aggregation models from a single-period to a multi-period setting. First, a formal model definition is given that is based on the rollover assumption of a constant level of risk. In a second step, more sophisticated risk management techniques are implemented. In Section 5, allocation techniques for the diversification benefit are presented. Section 6 concludes with a discussion of further improvements of risk type aggregation models.

2 Formal specification of the aggregation model

The objective of this section is to provide a formal definition of multi-factor aggregation models in a single-period setting.

Let $X = (X_1, \ldots, X_m)$ be a vector of real-valued random variables interpreted as risk factors at the planning horizon, e.g. 1 year. Typical examples of risk factors are macroeconomic factors as well as market factors such as equity indices or exchange rates. For each risk type $j = 1, \ldots, r$, let $H_j : \mathbb{R}^m \to \mathbb{R}$ be a deterministic function that derives losses from values of risk variables: the loss variable L_j of risk type j is specified by

$$L_j := H_j(X). \tag{1}$$

The aggregate loss variable L is defined by

$$L := \sum_j^r L_j. \tag{2}$$

In the following section, we present the model implemented at Deutsche Bank, a specific example of this class of multi-factor aggregation models.

3 Deutsche Bank's risk aggregation model

3.1 Identification of risk types and risk factors

Deutsche Bank calculates economic capital for the risk types credit risk, market risk, operational risk and business risk. Business risk is currently not considered in the risk aggregation model. It is planned to include this risk type as soon as realistic correlations to other risk types can be computed. For the time being, the economic capital for business risk is added to the diversified economic capital calculated in the aggregation model.

Five risk types are considered in the aggregation model: credit risk (C), operational risk (O) and market risk (M) excluding real estate risk (RE) and risk due to industrial holdings (IH), which are treated as separate risk types.

For each risk type $j \in \{\mathrm{C}, \mathrm{M}, \mathrm{RE}, \mathrm{IH}, \mathrm{O}\}$ there exists precisely one risk factor, denoted by X_j, that completely specifies the corresponding loss variable L_j. This assumption simplifies the definition of the loss functions H_j: each H_j can be considered as a univariate function $H_j : \mathbb{R} \to \mathbb{R}$ and the loss variable L_j of the j^{th} risk type is given by $L_j = H_j(X_j)$.

It is assumed that the risk factors X_j are standardized variables that follow a multi-dimensional normal distribution.[2] The specification of a realistic correlation matrix of these factors is a major challenge in the development of an aggregation model. Before we deal with this problem we will first define the loss functions H_j, $j \in \{\mathrm{C}, \mathrm{M}, \mathrm{RE}, \mathrm{IH}, \mathrm{O}\}$.

[2] The mean of a standardized variable is 0 and the variance is 1. The multi-dimensional distribution assumption will be discussed in Section 3.4.

3.2 Specification of loss functions

In Deutsche Bank's model, loss functions are derived from the stand-alone EC calculations for the different risk types. The EC figures for credit and operational risk are calculated via Monte Carlo simulation. In principle, the MC sample lists for these risk types could be used as representations of their loss functions. More precisely, the function H_j is then specified by

$$H_j(x) = Q_j(N(x)),$$

where N is the distribution function of a standardized normal distribution and $Q_j(\alpha)$ is the α-quantile of the discrete loss distribution given by the MC samples of risk type j. Under the assumption that the risk factor X_j of risk type j is a standardized normally distributed variable the loss distribution $L_j = H_j(X_j)$ equals the discrete distribution specified by the MC samples.

A different approach to the specification of the loss functions is based on analytic distribution families instead of discrete MC representations. Deutsche Bank uses the following parametric distribution families for the specification of the H_j: Vasicek distributions for credit risk (see, for example, Vasicek (2002) or Gordy (2003)), normal distributions for market risk and extreme value distributions for operational risk. The parametric distributions are fitted to the expected loss and the undiversified economic capital of each risk type. More precisely, the loss function H_C for credit risk has the form

$$H_\mathrm{C}(x) := l \cdot N(\frac{N^{-1}(p) + \sqrt{R^2}\, x}{\sqrt{1-R^2}}), \tag{3}$$

where the parameters l (portfolio exposure), p (average 1y default probability), R^2 (average R^2) are derived from Deutsche Bank's credit exposure and the expected loss and economic capital for credit risk. The extreme value distribution used for operational risk has the distribution function

$$F_{EV}(x) := \exp(-\lambda(1 + \frac{\xi}{\beta}x)^{-1/\xi}),$$

its α-quantile is denoted by $Q_{EV}(\alpha)$. The loss distribution H_O for operational risk is defined by

$$H_\mathrm{O}(x) := Q_{EV}(N(x)).$$

The parameter λ corresponds to the annual average frequency of OR losses in Deutsche Bank, the other two parameters are derived from expected loss and economic capital for operational risk. The specification of the loss functions for the different subtypes of market risk are straightforward since normal distribution assumptions are used for loss distributions as well as for risk factors: $H_j(x) := a_j x$ for $j \in \{\mathrm{M}, \mathrm{RE}, \mathrm{IH}\}$, where each multiplicative constant a_j is fitted to the economic capital of the corresponding risk type.

3.3 Estimation of correlations of risk factors

The correlations or, more precisely, the dependencies between the different risk types are the key variables influencing the scale of diversification benefits. Empirically, correlations can be measured by observing the long-run relationship between two data series. In practice, there is a limited amount of relevant data currently available for measuring correlations across risk types and there is little consistency across banks implementing economic capital methods regarding the manner in which these critical correlation values are selected (BIS, 2003; IFRI/CRO Forum survey, 2007). A popular technique is to measure correlations using data series that are considered as reasonable proxies for the underlying economic relationships. If no empirical estimates seem reliable, correlations are often specified in a judgemental way.

The Deutsche Bank model derives the correlations between credit and market risk factors from data series of credit and market risk proxies. The correlations between operational risk and the other risk types are derived from a qualitative analysis of high internal and external OR losses. Both methodologies will be reviewed in this section.

3.3.1 Estimation of correlations between credit and market risk factors

In Deutsche Bank, a structural credit portfolio model is used for the calculation of credit risk EC. It is based on 75 country and industry factors. Their covariance matrix is derived from time-series of corresponding MSCI equity indices. A weighted sum of these factors is used for the specification of the credit risk factor X_{C} in the risk aggregation model. More precisely, a time-series for the credit risk factor X_{C} is constructed as weighted sum of the MSCI time-series of the individual country and industry factors. The weights are determined by the composition of Deutsche Bank's credit portfolio. Hence, X_C can be interpreted as a global credit risk proxy for Deutsche Bank's portfolio, the credit loss distribution $L_{\mathrm{C}} = H_{\mathrm{C}}(X_{\mathrm{C}})$ used in the risk aggregation model is a homogeneous one-factor approximation of the inhomogeneous multi-factor model used in the stand-alone EC calculation for credit risk.

A similar construction is done for market risk: the risk factor X_{M} is a basket of credit spreads, interest rates and equity indices with portfolio weights obtained from the stress tests used to determine market risk EC. The risk factors X_{RE} and X_{IH} for real estate and industrial holdings are identified with a real estate index and an equity index respectively.

Since each of the risk factors X_{C}, X_{M}, X_{RE} and X_{IH} corresponds to an observable market factor or to a basket of observable market factors their correlation structure can be derived from the time-series of these factors.

3.3.2 Estimation of correlations for operational risk

In contrast to the other risk types, we have not identified an observable proxy for operational risk. A different strategy for determining correlations is used instead: correlation estimates between operational risk and the other risk types are based on a qualitative analysis of the largest internal OR losses as well as the largest relevant external losses. For each loss, a weight of 25%, 50% or 75% is assigned for low, medium or high dependence to credit and market risk respectively. A weight of 0% is assigned if the loss event is considered uncorrelated, 100% is assigned only in exceptional cases where credit or market risk explain virtually all of the loss amount. Generally, in the following situations weights greater than 0% are assigned:

1. CR or MR were drivers or one of the causes of the OR event (e.g. default of WorldCom or Enron),
2. OR caused CR or MR (e.g. 9/11),
3. CR or MR created the opportunity for the OR event (e.g. stock market bubble).

The total weights w_C and w_M for credit and market risk are then calculated as a weighted average of the weights of the individual events. Based on w_C and w_M, a risk factor X_O for OR is defined as a weighted sum of the risk factors for credit and market risk X_C and X_M and an independent risk factor X_I that is assumed standardized and normally distributed:

$$X_O = (w_C X_C + w_M X_M + (1 - w_C - w_M) X_I)/a, \tag{4}$$

where the scaling factor a ensures that the variance of X_O is 1. The correlations between X_O and the other risk factors X_j, $j \in \{\mathrm{C}, \mathrm{M}, \mathrm{RE}, \mathrm{IH}\}$ can now be derived from (4) and the correlation matrix of the credit and market factors.

3.4 The corresponding copula model

In the literature, there is usually a distinction between

1. multi-factor approach (base-level) and
2. aggregation through copulas (top level).

In order to analyze the relationship between these model classes we will first define the concept of a copula model.[3] For each risk type, let $L_1, \ldots, L_r$ specify the loss variables of the r risk types and denote their distribution functions by $F_1, \ldots, F_r$. A copula C is now used to combine the loss variables $L_1, \ldots, L_r$ to a r-dimensional loss vector $(L_1, \ldots, L_r)$: its distribution function is defined by

$$C(F_1(x_1), \ldots, F_r(x_r)). \tag{5}$$

Note that $F_1(x_1), \ldots, F_r(x_r)$ are the marginal distributions of (5). The aggregate loss variable L of $L_1, \ldots, L_r$ is now defined as the sum $L = L_1 + \ldots + L_r$.

It is an immediate consequence of Sklar's theorem that every multi-factor model of the form (2) can also be specified as a copula model. More precisely, let $F_1, \ldots, F_r$ be the distribution functions of the loss variables $H_1(X), \ldots, H_r(X)$ specified in (1). Then there exists a copula C such that $C(F_1(x_1), \ldots, F_r(x_r))$ is the distribution function of the loss vector $(H_1(X), \ldots, H_r(X))$. On the other hand, as long as no restriction is made on the risk factors and loss functions in the definition of the multi-factor model (2) every copula model can be formally defined in this way.

Hence, the purpose of separating risk aggregation models in multi-factor models and copula models is not the definition of distinct mathematical model classes. It rather emphasizes a conceptual difference in the construction process of these models: copula models are built by aggregating given loss variables for different risk types, whereas multi-factor models start with dependent systematic risk factors and define loss functions for different risk types on top.

Let Σ be the correlation matrix of the risk factors $X_1, \ldots, X_r$ used in the Deutsche Bank multi-factor model and denote the corresponding r-dimensional normal distribution and Gaussian copula by N_Σ and C_Σ respectively, i.e.

$$C_\Sigma(u_1, \ldots, u_r) := N_\Sigma(N^{-1}(u_1), \ldots, N^{-1}(u_r)).$$

We will now show that the Gaussian copula C_Σ specifies the joint distribution of the loss variables $(H_1(X_1), \ldots, H_r(X_r))$ used in Deutsche Bank's multi-factor model.

For all risk types, H_j is a strictly increasing function. Hence, $H_j(N^{-1}(\alpha))$ is the α-quantile of $H_j(X_j)$ and the distribution function of $H_j(X_j)$, denoted by F_j, is strictly increasing. It follows that the distribution function of $(H_1(X_1), \ldots, H_r(X_r))$ has the form

$$\begin{aligned}
\mathbb{P}(H_1(X_1) \le x_1, \ldots, H_r(X_r) \le x_r) &= \mathbb{P}(H_1(N^{-1}(N(X_1))) \le x_1, \ldots, \\
&\qquad\qquad H_r(N^{-1}(N(X_r))) \le x_r) \\
&= \mathbb{P}(N(X_1) \le F_1(x_1), \ldots, N(X_r) \le F_r(x_r)) \\
&= C_\Sigma(F_1(x_1), \ldots, F_r(x_r)),
\end{aligned}$$

[3]We refer to Nelson (1999) for an introduction to copulas.

which proves the claim.

The choice of the copula family usually has a significant impact on the aggregate loss variable L and, consequently, on the diversified economic capital. For example, since t-copulas have higher tail dependence than Gaussian copulas (McNeil et al., 2005) their application tends to reduce the diversification benefit. We have implemented t-copulas in Deutsche Bank's model and have recalculated economic capital using different degrees of freedom in the t-copula to assess the magnitude of the capital impact.

Given the limited amount of data currently available it seems questionable whether a reliable selection of a copula family can be achieved by purely statistical techniques. The choice of the copula therefore remains a source of uncertainty in the calculation of diversified economic capital as illustrated in the following example, see also Tang and Valdez (2006).

Example 1. We specify a hypothetical bank whose risk type aggregation model is based on the risk categories defined in Section 3.1: credit risk (C), market risk (M), real estate risk (RE), risk due to industrial holdings (IH) and operational risk (O). The following table provides the stand-alone economic capital for each of these risk types and the total (undiversified) EC, where for each risk type economic capital corresponds to the 99.98% quantile minus the expected loss.

	Risk Types					
in bn	C	M	RE	IH	O	Total
EC	4.0	3.5	0.25	0.25	2.0	10.0

Table 1: Undiversified economic capital

The marginal distributions are specified according to the methodology presented in Section 3.2. In order to illustrate the impact of the copula assumption we compare the results obtained from a Gaussian copula and a t-copula with 5 degrees of freedom. In both calculations the following correlation matrix is used:

$$\left(\begin{array}{c|ccccc} & C & M & RE & IH & O \\ \hline C & 1 & 0.7 & 0.4 & 0.7 & 0.3 \\ M & 0.7 & 1 & 0.4 & 0.7 & 0.3 \\ RE & 0.4 & 0.4 & 1 & 0.4 & 0.2 \\ IH & 0.7 & 0.7 & 0.4 & 1 & 0.3 \\ O & 0.3 & 0.3 & 0.2 & 0.3 & 1 \end{array}\right) \tag{6}$$

Note that in our example the correlations between the risk types credit, market and operational risk have the same magnitude as the correlations reported in the IFRI/CRO Forum survey (2007). Table 2 shows the results obtained under both copula assumptions.

First of all, this example confirms that the diversification benefit is very sensitive to the copula assumption: if a Gaussian copula is used the total EC is reduced from 10bn to 7.4bn whereas the t-copula only yields a reduction to 8.3bn. Hence, in this example the total diversification benefit (see (18)) is either 26% or 17% depending on the copula assumption.

The diversified ECs for the individual risk types are computed via Expected Shortfall allocation. More precisely, in a first step the confidence level $\alpha \in (0,1)$ is calculated such that

$$\mathrm{ES}_\alpha(L) = \mathrm{VaR}_{0.9998}(L),$$

in bn	Risk Types					Total
	C	M	RE	IH	O	
Undiversified EC	4.00	3.50	0.25	0.25	2.00	10.00
Gaussian copula	3.54	2.89	0.11	0.19	0.64	7.37
t-copula	3.68	3.18	0.14	0.21	1.07	8.28

Table 2: Diversified economic capital under different copula assumptions

where the value-at-risk $\mathrm{VaR}_{0.9998}(L)$ of the aggregate loss variable L at level 99.98% is simply the 99.98% quantile and the risk measure Expected Shortfall $\mathrm{ES}_\alpha(L)$ of L at level α is defined by[4]

$$\mathrm{ES}_\alpha(L) := (1-\alpha)^{-1} \int_\alpha^1 \mathrm{VaR}_u(L) du = E\left(L \mid L > \mathrm{VaR}_\alpha(L)\right). \tag{7}$$

In a second step, the diversified EC of risk type $j \in \{\mathrm{C}, \mathrm{M}, \mathrm{RE}, \mathrm{IH}, \mathrm{O}\}$ is computed by

$$\mathrm{E}\left(L_j \mid L > \mathrm{VaR}_\alpha(L)\right) - \mathrm{EL}(L_j),$$

where $\mathrm{EL}(L_j)$ specifies the expected loss of risk type j.

The diversified EC figures in Table 2 show that operational risk receives the highest diversification benefit. This is not surprising since it has the lowest correlations to the other risk types. Note, however, that the diversification benefit of the fat-tailed marginal OR loss distribution is also particularly sensitive to the copula assumption.

4 Risk type aggregation in a multi-period model

4.1 Rollover assumptions for liquid portfolios

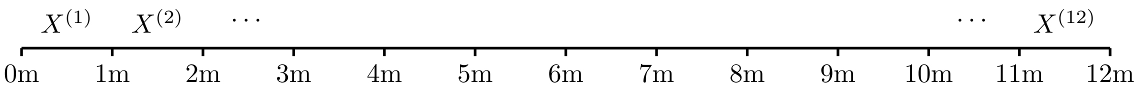
The aggregation model presented in the previous sections operates in a single-period framework: separate loss distributions are specified for the different risk types at the planning horizon, typically one year, and then aggregated in order to obtain the diversified risk capital. This setup does not work well for all risk types as illustrated by a comparison of risks in the banking book versus risks in the trading book. By definition, banking book positions are not actively traded and more difficult to hedge than trading book positions. Therefore, it seems justified to use a buy-and-hold assumption over the planning period of one year in the economic capital calculations for the banking book, e.g. for credit risk EC in the banking book. The same assumption would clearly overestimate the risk in the trading book: the higher liquidity of trading book positions facilitates the implementation of more effective risk management strategies.

Multi-period models provide the natural framework for the specification of portfolio strategies for liquid positions. In this setup, the planning period is split into time intervals determined by the liquidity horizon of the underlying portfolio, for example a 1y planning period might be split into 12 consecutive months. The rollover at the end of each period is specified by a portfolio strategy. In this subsection, we apply a simple strategy based on a constant level of risk. More precisely, it is assumed that the portfolio loss distribution at the beginning of each rollover period equals the initial loss distribution, e.g. the loss distribution for the first month also describes the potential losses in all consecutive months. The level of risk is kept constant in this portfolio strategy by rebalancing the portfolio at the end of each period. This strategy is in contrast to the assumption of holding the same position for the entire planning period of one year.

[4] Note that the equality in (7) holds because L has a continuous distribution.

Another basic model assumption is the independence of loss distributions in non-overlapping time periods.[5] This is a common assumption in risk modeling, which is additionally supported by the fact that rebalancing of the portfolio reduces potential autocorrelations, for example autocorrelations in rating migration processes. More precisely, under the assumption of a constant level of risk an asset is typically replaced if its credit quality has changed at the end of a liquidity period. Thus, the autocorrelations of rating migrations are not relevant for the aggregation of the loss distributions across non-overlapping time periods: in order to obtain a cumulative loss over one year, the loss distribution at the liquidity horizon is calculated and then re-sampled for the following time periods. For a portfolio with a liquidity horizon of 1m the 1y loss distribution therefore equals the convolution of 12 copies of the 1m loss distribution.

In order to formalize this strategy, we assume that the capital horizon is one year. We assign a liquidity horizon l_j to each of the r risk types and assume that $l_1, \ldots, l_r$ are elements of $\{1, 2, 3, 4, 6, 12\}$ corresponding to liquidity horizons of 1m, 2m, 3m, 4m, 6m and 12m.[6] Let $X^{(1)}, \ldots, X^{(12)}$ denote the monthly increments of all risk factors, which are assumed to be independent and identically distributed:



The risk vector X, which defines the status of all risk factors in one year, is the sum of the monthly increments

$$X = \sum_{s=1}^{12} X^{(s)}.$$

Loss variables for risk type j are specified by a loss function H_j together with the corresponding risk factors. Since we do not restrict H_j to a single risk factor in this section,

$$S_{j,t} := H_j \left(\sum_{s=t-l_j+1}^{t} X^{(s)} \right) \tag{8}$$

defines the loss variable covering a period of l_j months and ending in month t, where $t \in \{l_j, \ldots, 12\}$. We note the following observations:

1. The distribution of $S_{j,t}$ does not depend on t because of the i.i.d. assumption on the $X^{(s)}$. This implements the constant level of risk assumption.

2. S_{j,t_1} and S_{j,t_2} are independent if they cover non-overlapping intervals, i.e. if the sets $\{t_1 - l_j + 1, \ldots, t_1\}$ and $\{t_2 - l_j + 1, \ldots, t_2\}$ are disjoint.

3. For overlapping intervals, S_{j,t_1} and S_{j,t_2} are not independent (except for some special cases such as a constant loss function).

The 1y loss variable of the j^{th} risk type is now defined by

$$L_j := \sum_{t=1}^{12/l_j} S_{j,l_j \cdot t} \tag{9}$$

and therefore, under the assumption of constant level of risk formalized in (8),

$$L_j = \sum_{t=1}^{12/l_j} H_j \left(\sum_{s=l_j \cdot t - l_j + 1}^{l_j \cdot t} X^{(s)} \right). \tag{10}$$

[5] We refer to Dunn (2007) for a multi-period model with autocorrelations.

[6] These restrictions on liquidity horizons have been made to simplify the exposition and can be easily relaxed.

Note that definitions (1) and (9) agree if $l_j = 12$, that is, if the liquidity horizon equals the capital horizon of 12 months.

4.2 Aggregation across risk types

The rollover model described above can be easily extended to the aggregation of risk types with different liquidity horizons. For a capital horizon of one year, the aggregate loss across all risk types equals

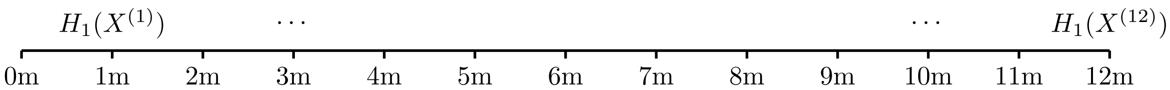
$$L := \sum_{j=1}^{r} L_j = \sum_{j=1}^{r} \sum_{t=1}^{12/l_j} S_{j,l_j \cdot t}. \tag{11}$$

Again, for a constant level of risk, we obtain

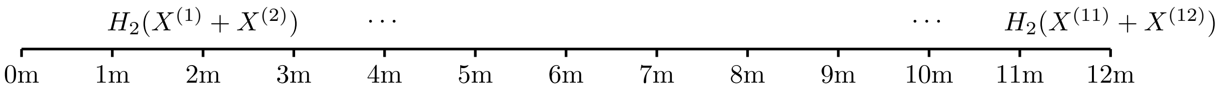
$$L = \sum_{j=1}^{r} \sum_{t=1}^{12/l_j} H_j \left(\sum_{s=l_j \cdot t - l_j + 1}^{l_j \cdot t} X^{(s)} \right).$$

The following charts illustrate this definition. In this example, it is assumed that the liquidity horizons of 3 risk types are 1 month, 2 months and 12 months. The losses are then calculated as shown below.

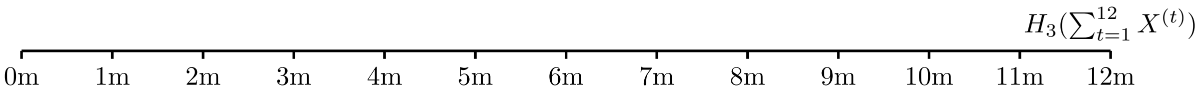
Losses for the first risk type with 1m liquidity horizon: $\sum_{t=1}^{12} H_1(X^{(t)})$



Losses for the second risk type with 2m liquidity horizon: $\sum_{t=1}^{6} H_2(X^{(2t-1)} + X^{(2t)})$



Losses for the third risk type with 12m liquidity horizon: $H_3(\sum_{t=1}^{12} X^{(t)})$



The aggregate loss across all risk types is given by

$$\sum_{t=1}^{12} H_1(X^{(t)}) + \sum_{t=1}^{6} H_2(X^{(2t-1)} + X^{(2t)}) + H_3(\sum_{t=1}^{12} X^{(t)}).$$

Note that the dependence structure of risk factors introduces dependence of the loss variables of the different risk types.

Example 2. We will now extend the single-period setup in Example 1 by modeling multi-period rollover strategies based on a constant level of risk. First of all, we define a multi-period model for the underlying risk factors: for $s = 1, \ldots, 12$ let

$$X^{(s)} := (X_{\mathrm{C}}^{(s)}, X_{\mathrm{M}}^{(s)}, X_{\mathrm{RE}}^{(s)}, X_{\mathrm{IH}}^{(s)}, X_{\mathrm{O}}^{(s)})$$

be random vectors that follow a multivariate normal distribution with mean $(0, \ldots, 0)$ and correlation matrix (6). Furthermore, assume that the standard deviation of each component $X_j^{(s)}$, $j \in \{\mathrm{C}, \mathrm{M}, \mathrm{RE}, \mathrm{IH}, \mathrm{O}\}$, equals $\sqrt{1/12}$ and that $X^{(1)}, \ldots, X^{(12)}$ are independent. Hence, the vector

$X^{(1)} + \ldots + X^{(12)}$ follows a multivariate normal distribution with covariance matrix (6), which is consistent with the Gaussian single-period model in Example 1.

We will now assume that market risk has a liquidity horizon of three months, whereas the 1y liquidity horizon is kept for the other risk types. In order to define a rollover strategy for market risk, a loss function H_{M} has to be specified that defines a loss distribution $H_{\mathrm{M}}(Q_{\mathrm{M}}^{(t)})$ for each 3m interval, where the 3m risk factors $Q_{\mathrm{M}}^{(t)}$ are defined by

$$Q_{\mathrm{M}}^{(t)} := X_{\mathrm{M}}^{(t-2)} + X_{\mathrm{M}}^{(t-1)} + X_{\mathrm{M}}^{(t)}, \qquad t = 3, \ldots, 12.$$

As in Section 3.2, we assume that H_{M} is of the form $H_{\mathrm{M}}(x) := a \cdot x$, where the multiplicative constant a is fitted to the market risk EC for the corresponding liquidity period. More precisely, a is chosen such that the 99.98% quantile of $H_{\mathrm{M}}(Q_{\mathrm{M}}^{(t)})$ equals the market risk EC for three months, denoted by $\mathrm{EC}_{3m}(\mathrm{M})$:[7]

$$\mathrm{EC}_{3m}(\mathrm{M}) = \mathrm{VaR}_{0.9998}(H_{\mathrm{M}}(Q_{\mathrm{M}}^{(t)})) = N^{-1}(0.9998) \cdot a \cdot \mathrm{StD}(Q_{\mathrm{M}}^{(t)}), \tag{12}$$

where StD denotes the standard deviation. For specifying $\mathrm{EC}_{3m}(\mathrm{M})$ we apply the popular scaling rule for market risk, which is based on the square-root of time: market risk EC for 1y is scaled by the factor $\sqrt{1/4} = 1/2$ to obtain market risk EC for 3 months. Since the standard deviation of $Q_{\mathrm{M}}^{(t)}$ is $1/2$ we immediately obtain from equality (12) that the scaling factor a equals the 1y EC divided by the 99.98% quantile of a standardized normally distributed variable, i.e.

$$a = \mathrm{EC}_{12\mathrm{m}}(\mathrm{M})/N^{-1}(0.9998) = 3.5/3.54\mathrm{bn}.$$

Note that precisely the same scaling factor a has been used in the definition of the loss function H_{M} applied in the single-period model. Hence, the aggregate loss variable

$$H_{\mathrm{M}}(Q_{\mathrm{M}}^{(3)}) + H_{\mathrm{M}}(Q_{\mathrm{M}}^{(6)}) + H_{\mathrm{M}}(Q_{\mathrm{M}}^{(9)}) + H_{\mathrm{M}}(Q_{\mathrm{M}}^{(12)})$$

of market risk, which is specified by rollover under a constant level of risk, equals the loss variable

$$H_{\mathrm{M}}(\sum_{s=1}^{12} X_{\mathrm{M}}^{(s)})$$

in the single-period model or, equivalently, the aggregate loss variable in the multi-period model if a liquidity horizon of one year is applied. As a consequence, the EC calculation in the current setup, i.e. 3m rollover for market risk and 1y liquidity horizon for all other risk types, produces results that are identical to the EC figures reported under the Gaussian copula assumption in Table 2.

Example 3. The above example illustrates that a rollover strategy based on a constant level of risk does not reduce economic capital if

1. loss distributions are Gaussian and
2. EC figures for different time periods are linked via the square-root of time.

We will now define a rollover model for credit risk that has a very different characteristic and leads to a significant reduction in EC. We use the same setup as in Example 2 but assume that 50% of the credit exposure has a liquidity horizon of one year, 25% has a liquidity horizon of six months and the remaining 25% is assigned to the three months liquidity bucket. Hence, according to the notation introduced in this section credit risk is split into three risk types that correspond to the different liquidity horizons. Separate loss functions H_{C12m}, H_{C6m} and H_{C3m} are specified for these risk types,

[7] The expected loss for market risk is assumed to be zero and therefore EC equals the 99.98% quantile.

however, we assume that each of these functions exclusively depends on the same credit risk factor, i.e. the 1y loss variables are defined by

$$L_{\mathrm{C12m}} := H_{\mathrm{C12m}}(\sum_{s=1}^{12} X_{\mathrm{C}}^{(s)}), \quad L_{\mathrm{C6m}} := \sum_{t=1}^{2} H_{\mathrm{C6m}}(\sum_{s=6t-5}^{6t} X_{\mathrm{C}}^{(s)}), \quad L_{\mathrm{C3m}} := \sum_{t=1}^{4} H_{\mathrm{C3m}}(\sum_{s=3t-2}^{3t} X_{\mathrm{C}}^{(s)}).$$

Each of the loss variables follows a Vasicek distribution and therefore the corresponding loss functions have the form

$$l \cdot N(\frac{N^{-1}(p) + \sqrt{R^2}\, x}{\sqrt{1-R^2}}),$$

see (3). The parameter l is defined by the credit exposure mapped to the corresponding liquidity bucket. The parameter p is interpreted as a default probability for a given liquidity horizon. In this example, the default probabilities for the six and three months liquidity horizons are obtained from the one year default probability by applying linear time-scaling factors of 1/2 and 1/4 respectively, see Bluhm et al. (2002) for more sophisticated scaling techniques based on rating migration matrices. The same R^2 is used in all loss functions.

Based on these specifications we obtain the following diversified EC figures presented in row *constant level of credit risk* of Table 3. The first two rows are given for comparison and present the undiversified capital figures and the diversified EC calculated in the single-period model with normally distributed risk factors (see Table 2).

	Risk Types					
in bn	C	M	RE	IH	O	Total
Undiversified EC	4.00	3.50	0.25	0.25	2.00	10.00
Single-period, Gaussian copula	3.54	2.89	0.11	0.19	0.64	7.37
Constant level of credit risk	2.61	2.97	0.11	0.19	0.68	6.55

Table 3: Constant level of credit risk

The diversified credit risk EC of 2.61bn is the sum of the capital figures calculated for the three credit risk types with liquidity horizons three, six and twelve months:

		Liquidity Horizons		
	Total	12m	6m	3m
Economic capital (in bn)	2.61	1.70	0.55	0.36
Credit exposure (in %)	100%	50%	25%	25%
Economic capital (in %)	100%	65%	21%	14%

Table 4: Constant level of credit risk: decomposition of EC

The large reduction in credit risk quantifies the difference in EC between a buy-and-hold portfolio and a strategy that rebalances the portfolio in order to keep the credit quality constant. A comparison of the last two columns in Table 4 illustrates that the EC reduction is significantly higher for shorter liquidity horizons: only 14% of the credit EC is allocated to the 3m liquidity bucket although it accounts for 25% of the credit exposure. This reduction is a consequence of the rebalancing strategy applied to the portfolio with 3m liquidity horizon. Whenever the initial rating of a transaction, say BBB, has changed at the end of a quarter it is replaced by another transaction with BBB rating and identical characteristics otherwise. Obviously, rebalancing reduces the number of defaults in portfolios of good credit quality, which explains the lower EC contribution of the 3m bucket compared to the buy-and-hold portfolio assigned to the 12m bucket.

4.3 Implementation of risk-reduction strategies

The constant level of risk formalizes a rather simplistic risk management strategy: the portfolio is rebalanced at each rollover and the portfolio size is kept constant over the entire planning period regardless of the loss history. It seems plausible, however, that the size of a portfolio will be reduced after suffering high losses. In order to facilitate the implementation of portfolio strategies other than a constant level of risk we have to extend the aggregation model.

A class of more sophisticated portfolio strategies is based on the definition of a function $M_j : \mathbb{R} \to \mathbb{R}$ for each risk type j that scales the size of the portfolio depending on the loss history. More formally,

$$\begin{aligned} S_{j,l_j} &:= H_j\left(\sum_{s=1}^{l_j} X^{(s)}\right), \\ S_{j,l_j \cdot t} &:= M_j\left(\sum_{s=1}^{t-1} S_{j,l_j \cdot s}\right) \cdot H_j\left(\sum_{s=l_j \cdot t - l_j + 1}^{l_j \cdot t} X^{(s)}\right) \qquad \text{for } t = 2, \ldots, 12/l_j. \end{aligned} \tag{13}$$

The loss distribution $S_{j,l_j \cdot t}$ of the t^{th} rollover period is scaled by M_j, where values above 1 increase the portfolio size, values between 0 and 1 correspond to an exposure reduction and negative values imply "going short". The function M_j is applied to the aggregated loss history $\sum_{s=1}^{t-1} S_{j,l_j \cdot s}$ in the previous rollover periods. Note, however, that the sum $\sum_{s=1}^{t-1} S_{j,l_j \cdot s}$ can be easily replaced as argument of M_j by other functions of $S_{j,l_j}, \ldots, S_{j,l_j \cdot (t-1)}$, e.g. by the maximum loss $\max(S_{j,l_j}, \ldots, S_{j,l_j \cdot (t-1)})$ observed in the previous $t-1$ rollovers or simply by the loss $S_{j,l_j \cdot (t-1)}$ in the most recent rollover period.

In line with (9) and (11), the annual loss for the j^{th} risk type and the aggregate annual loss across all risk types are defined by

$$L_j := \sum_{t=1}^{12/l_j} S_{j,l_j \cdot t}, \qquad L := \sum_{j=1}^{r} L_j. \tag{14}$$

We now consider specific risk management strategies, first the cases $M_j = 1$ and $M_j = 0$.

1. The definition $M_j = 1$ corresponds to a constant level of risk: in this case the definition (13) specializes to (8) and the annual loss equals

$$L_j = \sum_{t=1}^{12/l_j} H_j\left(\sum_{s=l_j \cdot t - l_j + 1}^{l_j \cdot t} X^{(s)}\right).$$

2. The definition $M_j = 0$ corresponds to investment in risk-free assets. More precisely, the current portfolio is sold at liquidity horizon and the money is invested in risk-free assets for the rest of the year. This effectively means that the capital horizon is set to the liquidity horizon. Under this assumption, the annual aggregate loss equals

$$L_j = S_{j,l_j} = H_j\left(\sum_{s=1}^{l_j} X^{(s)}\right).$$

3. We now specify a more sophisticated portfolio strategy. In this example, the size of the portfolio is reduced in case of high aggregate losses:

$$M_j(a) := 1 \text{ for } a \leq A, \qquad M_j(a) := q \text{ for } a > A, \tag{15}$$

where A is a positive real number and q is a number between 0 and 1. The function M_j has the following interpretation. A threshold for the aggregate loss amount is specified by A. At the beginning of each rollover period it is checked whether A is exceeded by the aggregate losses in previous rollover periods. In this case, the size of the portfolio is reduced, where the downscaling factor is specified by q.

Example 4. It has been shown in Example 2 that there is no impact on market risk EC if market risk is rolled over based on a constant level of risk. We will now implement a risk-reduction strategy for market risk that is based on a limit for aggregate losses using the formal framework specified in (15). More precisely, the market risk exposure is reduced by 50% if the aggregate loss exceeds the 90% quantile of the (stand-alone) 1y loss distribution for market risk, i.e., the parameters in (15) are set to

$$A := \frac{N^{-1}(0.9)}{N^{-1}(0.9998)} \cdot 3.5\text{bn} = 1.267\text{bn}, \qquad q := 0.5$$

with a liquidity horizon of three months. In other words, if an aggregate market risk loss of more than 1.267bn is observed at the end of any quarter the market risk exposure is reduced to 50% in the next quarter. In this example it is assumed that all other risk types have a liquidity horizon of one year and their loss functions are specified as in the examples 1 and 2. The following table shows the results of the calculation in row *loss limit for market risk*, whereas the first two rows present the undiversified capital figures and the diversified EC calculated in the single-period model with normally distributed risk factors for comparison (see Table 2).

in bn	Risk Types					Total
	C	M	RE	IH	O	
Undiversified EC	4.00	3.50	0.25	0.25	2.00	10.00
Single-period, Gaussian copula	3.54	2.89	0.11	0.19	0.64	7.37
Loss limit for market risk	3.65	2.29	0.11	0.18	0.63	6.87

Table 5: Loss limit for market risk

The exposure reduction by 50% in case of a breach of the loss limit results in a decrease of the diversified EC for market risk by 0.6bn, whereas the total EC is reduced by 0.5bn.

The class of rollover strategies discussed above is implemented on portfolio level. More precisely, the portfolio size is adjusted in each rollover depending on the performance of the entire portfolio in previous rollover periods. We will conclude this section with a rollover strategy relevant for credit risk management, which makes adjustments on asset level.

Note that under the assumption of a constant level of risk, the defaulted part of a portfolio is replaced by performing assets in each rollover period. As a consequence, an asset (together with its replacements in future rollovers) may default several times within the 12 months planning period. A shorter liquidity horizon may therefore lead to a higher risk capital. This problem is particularly pronounced for subinvestment grade portfolios. In the following, we will formalize a rollover strategy, where each position can only default once and therefore this non-intuitive behavior is avoided.

Let l_j be the liquidity horizon of risk type j and assume that the underlying portfolio consists of n assets.[8] More precisely, assume that the loss function H_j of risk type j can be written as $H_j = \sum_{i=1}^{n} H_{ji}$, where H_{ji} is the loss function of the i^{th} asset. Equation (10) expresses the portfolio loss under the assumption of a constant level of risk. It can be rewritten in terms of the H_{ji}:

$$L_j = \sum_{t=1}^{12/l_j} H_j\left(\sum_{s=l_j\cdot t-l_j+1}^{l_j\cdot t} X^{(s)}\right) = \sum_{t=1}^{12/l_j}\sum_{i=1}^{n} H_{ji}\left(\sum_{s=l_j\cdot t-l_j+1}^{l_j\cdot t} X^{(s)}\right). \tag{16}$$

For each asset $i = 1, \dots, n$, we define a survival indicator I_{ji} that covers periods of l_j months: for every $t \in \{l_j, \dots, 12\}$

$$\begin{aligned} I_{ji}\left(\sum_{s=t-l_i+1}^{t} X^{(s)}\right) &:= 0 \quad \text{if asset } i \text{ defaults between months } t-l_i+1 \text{ and } t,\\ I_{ji}\left(\sum_{s=t-l_i+1}^{t} X^{(s)}\right) &:= 1 \quad \text{if asset } i \text{ survives in this period.} \end{aligned}$$

By taking products over consecutive periods of l_j months, we obtain a survival indicator for the first $t = 1, \dots, 12/l_j$ rollover periods, i.e., the indicator

$$\prod_{u=1}^{t} I_{ji}\left(\sum_{s=l_j\cdot u-l_j+1}^{l_j\cdot u} X^{(s)}\right) \tag{17}$$

equals 1 if and only if the i^{th} asset survives in each of the t rollover periods up to month $t \cdot l_j$. The combination of (16) and (17) yields the following definition of the portfolio loss at the planning horizon:

$$L_j := \sum_{t=1}^{12/l_j}\sum_{i=1}^{n} H_{ji}\left(\sum_{s=l_j\cdot t-l_j+1}^{l_j\cdot t} X^{(s)}\right)\cdot\prod_{u=1}^{t-1} I_{ji}\left(\sum_{s=l_j\cdot u-l_j+1}^{l_j\cdot u} X^{(s)}\right)$$

specifies the 1y loss variable for risk type j under the assumption of a constant level of risk without multiple defaults. In other words, this rollover strategy assumes that

1. if an asset has defaulted in rollover period t it is removed from the portfolio in the remaining rollovers,
2. the non-defaulted part of the portfolio is adjusted at the beginning of each rollover period according to the assumption of a constant level of risk.

5 Calculation and allocation of diversified EC

5.1 Diversification benefit

The objective of the previous sections was the introduction of a framework for aggregating loss distributions across risk types. This section deals with the actual calculation and allocation of the diversification benefit.

Monte Carlo techniques are typically used for simulating the aggregate loss distribution across risk types in single-period as well as multi-period models.[9] Since the aggregate loss distribution L is the

[8] As in previous sections, we restrict ourselves to a homogenous liquidity horizon within risk type j, i.e., each of the n assets has liquidity horizon l_j.

[9] Rüß (2008) analyzes the numerical stability of MC techniques in risk aggregation models and proposes low-discrepancy sequences as alternative computation technique.

sum of the loss distributions L_j of the individual risk types, i.e. $L = \sum L_j$, standard risk measures like Value-at-Risk or Expected Shortfall and their corresponding allocation techniques can be used for calculating economic capital and allocating it to the different risk types. The diversification benefit for risk type j is quantified by comparing its contributory (diversified) EC in the risk aggregation model, denoted by $\mathrm{DEC}(j)$, and its standalone economic capital $\mathrm{EC}(j)$. More formally, the factor

$$1 - \mathrm{DEC}(j)/\mathrm{EC}(j) \tag{18}$$

specifies the diversification benefit for risk type j. If the calculations of $\mathrm{EC}(j)$ and $\mathrm{DEC}(j)$ are based on a coherent risk measure (Artzner et al., 1999) and the corresponding allocation (Kalkbrener, 2005) then $\mathrm{DEC}(j) \leq \mathrm{EC}(j)$. Hence, the diversification benefit (18) is always non-negative. Note that this property does not necessarily hold for Value-at-Risk, see Embrechts et al. (2005) for an analysis.

Similarly to (18), the diversification benefit across all risk types is determined by the quotient of the economic capital DEC in the risk aggregation model and the sum of the standalone EC figures of the individual risk types, i.e. the diversification benefit is given by

$$1 - \mathrm{DEC}/(\mathrm{EC}(1) + \ldots + \mathrm{EC}(r)).$$

5.2 Allocation of the diversification benefit to business units

The same techniques can be used for allocating diversified economic capital to business units if risk types are modeled separately for each unit, i.e. L_j can be decomposed into $L_j = \sum L_{jk}$, where L_{jk} is the loss variable that specifies the loss of risk type j in business unit k. If this information is not available the allocation of the diversification benefit to business units is usually done proportionally to their contributions to standalone risk type EC. More precisely, let $\mathrm{EC}(\mathrm{BU}, j)$ be the contributory EC of business unit BU with respect to risk type j. Then the diversified EC contribution of the business unit is defined by

$$\mathrm{DEC}(\mathrm{BU}, j) := \mathrm{EC}(\mathrm{BU}, j) \cdot \frac{\mathrm{DEC}(j)}{\mathrm{EC}(j)}. \tag{19}$$

Note that the diversification factor $\mathrm{DEC}(j)/\mathrm{EC}(j)$ for risk type j determines the diversification benefit for each business unit with respect to risk type j. Furthermore, definition (19) preserves the additivity of EC contributions, i.e.

$$\mathrm{EC}(j) = \sum_{\mathrm{BU}} \mathrm{EC}(\mathrm{BU}, j) \quad \text{implies} \quad \mathrm{DEC}(j) = \sum_{\mathrm{BU}} \mathrm{DEC}(\mathrm{BU}, j).$$

Apart from methodological problems, the allocation of the diversification benefit to business units is a controversial conceptual issue in the finance industry (BIS, 2003). Some banks are adamant that business unit economic capital should be calculated on a stand-alone basis, with any diversification benefit accruing only at the firm-wide level. These banks contend that business unit decision-making could be distorted if activities are assumed to be low risk only because of the existence of diversification benefits with other business units. Other banks, however, do apportion the diversification benefits down to individual business units. They believe that such measures provide a more accurate reflection of a unit's marginal contribution to firm-wide risk and that concerns regarding inappropriate incentives can be addressed through the risk management process.

6 Future developments

In the first part of this paper, a single-period model for risk type aggregation has been presented. This model is a useful tool for quantifying the correct magnitude of diversification effects. However, due to its simplicity it does not provide an appropriate framework for a granular and more accurate aggregation of risk types. A particular shortcoming is the joint modeling of risk types with different liquidity characteristics, e.g. the joint modeling of credit and market risk. In a recent working paper, the Basel Committee identified the different horizons for risk measurement, which are largely caused by different liquidity profiles, as the most important obstacle to the further integration of market and credit risk assessments, see Basel Committee on Banking Supervision (2009). The importance of market liquidity for credit and market risk management has also been highlighted during the current financial crisis. A prominent example is the downward spiral between market prices and liquidity, which has been experienced in structured product markets. Going forward, aggregation models are needed that reflect the impact of market liquidity on other risk types.

As a first step to resolve this problem we have proposed a multi-period model, which improves the aggregation of risk types with different liquidity horizons and allows for the implementation of risk management strategies. Its actual application in the EC process requires a liquidity analysis of the relevant portfolios and the specification of rollover strategies.

The aggregation model can be further improved by replacing the one-dimensional risk type proxies by multi-dimensional risk factors, for example multiple factors for credit and market risk. A common dependence structure has to be specified for these factors, which provides the basis for the multi-period simulation of such an integrated risk model. Sophisticated variance reduction techniques are usually required in the simulation process in order to cope with the inherent computational complexity of the model.

In our opinion, the development and implementation of advanced models for risk type aggregation will be a major topic in quantitative risk management, despite all the difficulties listed above. The main reason is that a realistic assessment of many risks facing a financial institution cannot be split into separate calculations in the classical stand-alone models for credit, market and operational risk. Instead, a quantitative framework is needed that provides an integrated view on the risk profile of a bank including assets, liabilities and off balance sheet items, see, for example, Drehmann et al. (2008).

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