An axiomatic approach to capital allocation

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Abstract

Capital allocation techniques are of central importance in portfolio management and risk-based performance measurement. In this paper we propose an axiom system for capital allocation and analyze its satisfiability and completeness: it is shown that for a given risk measure ρ there exists a capital allocation Λ_ρ which satisfies the main axioms if and only if ρ is sub-additive and positively homogeneous. Furthermore, it is proved that the axiom system uniquely specifies Λ_ρ. We apply the axiomatization to the most popular risk measures in the finance industry in order to derive explicit capital allocation formulae for these measures.

Key Words: capital allocation, risk measure, expected shortfall, value-at-risk, Hahn-Banach theorem

1 Introduction

The application of risk measures in portfolio management or performance measurement requires the allocation of risk capital either to subportfolios or to business units. More formally, assume that a risk measure ρ has been fixed and let X be a portfolio which consists of subportfolios $X_1, \ldots, X_m$, i.e. $X = X_1 + \ldots + X_m$. The objective is to distribute the risk capital $k := \rho(X)$ of the portfolio X to its subportfolios, i.e. to compute risk contributions $k_1, \ldots, k_m$ of $X_1, \ldots, X_m$ with $k = k_1 + \ldots + k_m$.

Allocation problems have been extensively studied in game theory. In recent years the banking industry recognized the importance of allocation techniques. Theoretical and practical aspects of different allocation schemes have been analyzed in a number of papers, for instance in Garman (1996, 1997), Hallerbach (1999), Schmock and Straumann (1999), Artzner et al. (1999b), Delbaen (2000), Overbeck (2000), Denault (2001), Tasche (2000, 2002), Fischer (2003) and Urban et al. (2003).

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In this paper we propose a simple axiomatization of capital allocation. It is based on the assumption that the capital allocated to subportfolio X_i only depends on X_i and X but not on the decomposition of the rest

$$X - X_i = \sum_{j \neq i} X_j$$

of the portfolio. Hence, a capital allocation Λ can be considered as a function of two arguments, the first being the subportfolio X_i and the second the portfolio X. Λ is called a capital allocation with respect to a risk measure ρ if it satisfies $\Lambda(X, X) = \rho(X)$, i.e. the capital allocated to X (considered as stand-alone portfolio) is the risk capital $\rho(X)$ of X. We propose a system of three axioms for capital allocation: linear aggregation, diversification and continuity at X. The first ensures that the sum of the risk capital of the subportfolios equals the risk capital of the portfolio, the second formalizes diversification and the last ensures that small changes to a portfolio X only have a limited effect on the risk capital of its subportfolios.

Despite its simplicity this axiom system uniquely characterizes capital allocation, i.e. for a given risk measure there exists at most one capital allocation which satisfies the axiom system. We show that the uniquely determined risk capital of X_i considered as subportfolio of X is the derivative of the underlying risk measure ρ at X in direction of subportfolio X_i, in agreement with results in the papers cited above.

After characterizing capital allocation schemes we turn to their existence. By applying the Hahn-Banach theorem we show that for a given risk measure ρ there exists a capital allocation Λ_ρ which satisfies the linear aggregation and diversification axioms if and only if ρ is sub-additive and positively homogeneous, i.e. ρ satisfies

$$\rho(X+Y) \;\leq\; \rho(X) + \rho(Y) \quad \text{and} \quad \rho(aX) \;=\; a\rho(X) \text{ for } a \geq 0.$$

Furthermore, we prove that the existence of all directional derivatives of ρ at a portfolio X is a necessary and sufficient condition for Λ_ρ being continuous at X.

Value-at-risk, standard deviations and expected shortfall are the most popular risk measures in the finance industry. The development of sound capital allocation techniques for these measures is an important practical problem. In contrast to value-at-risk, expected shortfall and risk measures based on standard deviations are sub-additive and positively homogeneous. For these two classes we derive explicit formulae which specify linear, diversifying capital allocations. We finish the paper with a discussion of different allocation schemes for value-at-risk.

2 An axiom system for capital allocation

In this paper let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, L^0 the space of all equivalence classes of real valued random variables on Ω and V a subspace of the vector space L^0.

We will identify each portfolio X with its loss function, i.e. X is an element of V and $X(\omega)$ specifies the loss of X at a future date in state $\omega \in \Omega$. We assume that a function $\rho : V \to \mathbb{R}$ has been defined. For each $X \in V$, $\rho(X)$ specifies the risk capital associated with portfolio X. At this point we do not require that the risk measure ρ has specific properties.

Let $X \in V$ be a portfolio which consists of subportfolios $X_1, \ldots, X_m \in V$, i.e. $X = X_1 + \ldots + X_m$. We want to distribute the risk capital of the portfolio $k = \rho(X)$ to its subportfolios. Denote the amount of capital allocated to subportfolio X_i by k_i. The proposed axiomatization is based on the assumption that the capital k_i only depends on X_i and X but not on the decomposition of the rest

$$X - X_i = \sum_{j \neq i} X_j.$$

More formally, let $X = Y_1 + \ldots + Y_n$ be another decomposition of X with capital allocation $l_1, \ldots, l_n$. If $X_i = Y_j$ for indices i, j then $k_i = l_j$.

Based on the above assumption a capital allocation Λ can be defined as a function from $V \times V$ to $\mathbb{R}$. Its meaning is that $\Lambda(X_i, X)$ defines the capital allocated to X_i if X_i is considered a subportfolio of portfolio X. Λ is called a capital allocation with respect to the risk measure ρ if it satisfies $\Lambda(X, X) = \rho(X)$, i.e. the capital allocated to X (considered as stand-alone portfolio) is the risk capital $\rho(X)$ of X. Furthermore, we propose the following axioms for Λ:

1. **Linear aggregation:** The risk capital of the portfolio equals the sum of the (contributory) risk capital of its subportfolios. More formally, let $X_1, \ldots, X_m \in V$ and $a_1, \ldots, a_m \in \mathbb{R}$ and define $X = a_1 X_1 + \ldots + a_m X_m$. Then
$$\rho(X) = \Lambda(X, X) = \sum_{i=1}^{m} a_i \Lambda(X_i, X).$$

2. **Diversification:** The risk capital $\Lambda(X, Y)$ of $X \in V$ considered as a subportfolio of $Y \in V$ does not exceed the risk capital $\rho(X) = \Lambda(X, X)$ of X considered as stand-alone portfolio.

3. **Continuity:** Small changes to the portfolio only have a limited effect on the risk capital of its subportfolios. More formally, the risk capital $\Lambda(X, Y + \epsilon X)$ of X in $Y + \epsilon X$ converges to the risk capital $\Lambda(X, Y)$ of X in Y if $\epsilon \in \mathbb{R}$ converges to 0.

The following definition formalizes these principles (see Denault (2001) for an alternative axiomatic approach).

DEFINITION 2.1. Let $\rho : V \to \mathbb{R}$. A capital allocation (with respect to ρ) is a function Λ from $V \times V$ to $\mathbb{R}$ such that for every $X \in V$

$$\Lambda(X, X) = \rho(X).$$

The capital allocation Λ is called

$$\begin{array}{lrcll} \textit{linear:} & \Lambda(aX+bY,Z) & = & a\Lambda(X,Z)+b\Lambda(Y,Z) & \forall a,b\in\mathbb{R},\ X,Y,Z\in V, \\ \textit{diversifying:} & \Lambda(X,Y) & \leq & \Lambda(X,X) & \forall X,Y\in V. \end{array}$$

Let $Y \in V$. The capital allocation Λ is called

$$\textit{continuous at } Y\textit{:} \quad \lim_{\epsilon\to 0}\Lambda(X,Y+\epsilon X)=\Lambda(X,Y) \qquad \forall X\in V.$$

We will now analyze this axiomatization. In particular, we will deal with the following questions:

1. **Completeness:** For a given ρ, does the axiom system uniquely determine capital allocations? If not, does it make sense to add further axioms to the system?
2. **Existence:** What are necessary and sufficient properties of risk measures to ensure existence of capital allocations which satisfy the axioms?
3. **Allocation formulae:** Can we explicitly specify capital allocations for particular classes of risk measures?

3 Completeness of the axiom system

It is interesting that this simple axiom system is already complete: if a linear, diversifying capital allocation is continuous at a portfolio $Y \in V$ then the risk capital $\Lambda(X,Y)$ of an arbitrary subportfolio X is uniquely determined. It is the derivative of the underlying risk measure ρ at Y in direction of subportfolio X.

THEOREM 3.1. *Let Λ be a linear, diversifying capital allocation with respect to ρ. If Λ is continuous at $Y \in V$ then for all $X \in V$*

$$\Lambda(X,Y)=\lim_{\epsilon\to 0}\frac{\rho(Y+\epsilon X)-\rho(Y)}{\epsilon}.$$

Proof. Let $\epsilon, \bar{\epsilon} \in \mathbb{R}$. Since Λ is linear and diversifying,

$$\begin{aligned} \rho(Y+\bar{\epsilon}X) &\geq \Lambda(Y+\bar{\epsilon}X, Y+\epsilon X) \\ &= \Lambda((Y+\epsilon X)+(\bar{\epsilon}-\epsilon)X, Y+\epsilon X) \\ &= \rho(Y+\epsilon X)+(\bar{\epsilon}-\epsilon)\Lambda(X,Y+\epsilon X). \end{aligned}$$

If $\epsilon < \bar{\epsilon}$ then

$$\Lambda(X,Y+\epsilon X)\leq\frac{\rho(Y+\bar{\epsilon}X)-\rho(Y+\epsilon X)}{\bar{\epsilon}-\epsilon}\leq\Lambda(X,Y+\bar{\epsilon}X). \tag{3.1}$$

Since Λ is continuous at Y,

$$\Lambda(X,Y) = \lim_{\epsilon \to 0} \frac{\rho(Y+\epsilon X) - \rho(Y)}{\epsilon}. \quad \square$$

4 Existence of capital allocations

DEFINITION 4.1. The risk measure $\rho : V \to \mathbb{R}$ is called

$$\begin{array}{llcll} \textit{positively homogeneous:} & \rho(aX) & = & a\rho(X) & \forall a \geq 0,\ X \in V, \\ \textit{sub-additive:} & \rho(X+Y) & \leq & \rho(X)+\rho(Y) & \forall X, Y \in V. \end{array}$$

In this section we will show that there exists a linear, diversifying capital allocation Λ with respect to ρ if and only if ρ is positively homogeneous and sub-additive. The proof of this theorem is an application of one of the fundamental results in functional analysis, the Hahn-Banach theorem (see, for instance, Theorem II.3.10 in Dunford and Schwartz (1958)).[1]

First we will show that ρ can be represented in the form

$$\rho(X) = \max\{h(X) \mid h \in H\},$$

where H consists of real valued, linear functions on V.

DEFINITION 4.2. Let V^* be the set of real valued, linear functions on V and

$$H_\rho := \{h \in V^* \mid h(X) \leq \rho(X) \text{ for all } X \in V\}.$$

THEOREM 4.1. *Let $\rho : V \to \mathbb{R}$ be a positively homogeneous and sub-additive risk measure. Then*

$$\rho(X) = \max\{h(X) \mid h \in H_\rho\} \tag{4.1}$$

for all $X \in V$.

Proof. Let $Y \in V$ and V_Y the linear subspace of V generated by Y. Define f_Y on V_Y by $f_Y(aY) := a \cdot \rho(Y)$ for every real number a. Since $\rho(Y) + \rho(-Y) \geq \rho(0) = 0$ it follows that $\rho(-Y) \geq -\rho(Y)$. Since ρ is positively homogeneous,

$$f_Y(aY) = \rho(aY), \quad f_Y(-aY) = -a \cdot \rho(Y) \leq a \cdot \rho(-Y) = \rho(-aY) \qquad \text{for } a \geq 0.$$

[1] The results in this section can also be shown in the framework of convex analysis. More precisely, since every sub-additive and positively homogeneous function ρ is convex the results can be derived from theorems on subgradients of convex functions.

Hence, $f_Y \leq \rho$ on V_Y. It follows from the Hahn-Banach theorem that there exists an $h_Y \in V^*$ such that

$$h_Y(X) = f_Y(X) \text{ for all } X \in V_Y, \qquad h_Y(X) \leq \rho(X) \text{ for all } X \in V.$$

Hence, $h_Y \in H_\rho$ with $h_Y(Y) = \rho(Y)$ and (4.1) is proved. □

We can now define the following capital allocation with respect to ρ.

DEFINITION 4.3. Let ρ be a positively homogeneous and sub-additive risk measure. For every $Y \in V$ let h_Y be an element of H_ρ with $h_Y(Y) = \rho(Y)$ and define for every $X, Y \in V$

$$\Lambda_\rho(X, Y) := h_Y(X). \tag{4.2}$$

The existence of an element h_Y of H_ρ with $h_Y(Y) = \rho(Y)$ has been shown in the proof of Theorem 4.1. Note, however, that h_Y is not necessarily unique: there exists a unique $h_Y \in H_\rho$ with $h_Y(Y) = \rho(Y)$ if and only if the directional derivative

$$\lim_{\epsilon \to 0} \frac{\rho(Y + \epsilon X) - \rho(Y)}{\epsilon}$$

exists for every $X \in V$ (see Theorem 4.3).

In the examples in section 5, the functions h_Y are not only linear but also continuous. The reason is that in these examples the linear space V is equipped with a norm $\|.\|$ and ρ is bounded, i.e.

$$\sup_{\|X\| \leq 1} |\rho(X)| < \infty.$$

Therefore, $h_Y \leq \rho$ implies that $\Lambda_\rho(., Y) = h_Y$ is bounded and therefore continuous with respect to the topology induced by the norm (see, for instance, Lemma II.3.4 in Dunford and Schwartz (1958)). Hence, the set V^* of real valued, linear functions on V (Definition 4.2) can be replaced by the topological dual of V in the examples.

THEOREM 4.2.

(a) *If there exists a linear, diversifying capital allocation Λ with respect to ρ then ρ is positively homogeneous and sub-additive.*
(b) *If ρ is positively homogeneous and sub-additive then Λ_ρ is a linear, diversifying capital allocation with respect to ρ.*

Proof. (a) Note that for $a \geq 0$

$$a\Lambda(X, X) = \Lambda(aX, X) \leq \Lambda(aX, aX) = a\Lambda(X, aX) \leq a\Lambda(X, X).$$

Hence,

$$a\rho(X) = a\Lambda(X, X) = \Lambda(aX, aX) = \rho(aX)$$

and ρ is positively homogeneous. By

$$\rho(X + Y) = \Lambda(X, X + Y) + \Lambda(Y, X + Y) \leq \Lambda(X, X) + \Lambda(Y, Y) = \rho(X) + \rho(Y),$$

ρ is sub-additive.

(b) It immediately follows from (4.1) and (4.2) that Λ_ρ is a linear, diversifying capital allocation with respect to ρ. $\square$

If ρ is positively homogeneous and sub-additive then Λ_ρ is a linear, diversifying capital allocation with respect to ρ. We will now analyze under which conditions the capital allocation Λ_ρ satisfies the continuity axiom

$$\Lambda_\rho \textit{ is continuous at } Y \in V: \quad \lim_{\epsilon \to 0} \Lambda_\rho(X, Y + \epsilon X) = \Lambda_\rho(X, Y) \qquad \forall X \in V.$$

First of all, it follows from Theorems 3.1 and 4.2 that the existence of all directional (or Gateaux) derivatives of ρ at $Y \in V$ is a necessary condition for the continuity of Λ_ρ at Y. The following theorem shows that this condition is also sufficient. A third equivalent condition is the uniqueness of $h \in H_\rho$ with $h(Y) = \rho(Y)$ (which is a well-known characterization of the differentiability of a convex function).

THEOREM 4.3. *Let ρ be a positively homogeneous and sub-additive risk measure and $Y \in V$. Then the following three conditions are equivalent:*
(a) *Λ_ρ is continuous at Y, i.e. for all $X \in V$*

$$\lim_{\epsilon \to 0} \Lambda_\rho(X, Y + \epsilon X) = \Lambda_\rho(X, Y).$$

(b) *The directional derivative*

$$\lim_{\epsilon \to 0} \frac{\rho(Y + \epsilon X) - \rho(Y)}{\epsilon}$$

exists for every $X \in V$.
(c) *There exists a unique $h \in H_\rho$ with $h(Y) = \rho(Y)$.*

If these conditions are satisfied then

$$\Lambda_\rho(X, Y) = \lim_{\epsilon \to 0} \frac{\rho(Y + \epsilon X) - \rho(Y)}{\epsilon} \tag{4.3}$$

for all $X \in V$.

Proof. (a) $\Rightarrow$ (b) and equality (4.3) follow from Theorems 3.1 and 4.2.

(b) $\Rightarrow$ (c) : Let $h \in H_\rho$ with $h(Y) = \rho(Y)$. Then, for every $\epsilon \in \mathbb{R}$ and $X \in V$,

$$\rho(Y + \epsilon X) - \rho(Y) \geq h(Y + \epsilon X) - h(Y) = \epsilon h(X).$$

Therefore,

$$\lim_{\epsilon \to 0-} \frac{\rho(Y + \epsilon X) - \rho(Y)}{\epsilon} \leq h(X) \leq \lim_{\epsilon \to 0+} \frac{\rho(Y + \epsilon X) - \rho(Y)}{\epsilon}$$

and uniqueness of h follows from the equality of these limits.
(c) $\Rightarrow$ (a) : Let $X \in V$. First of all, note that

$$\rho(Y) - \rho(-\epsilon X) = \rho(Y + \epsilon X - \epsilon X) - \rho(-\epsilon X) \leq \rho(Y + \epsilon X) \leq \rho(Y) + \rho(\epsilon X)$$

implies

$$\lim_{\epsilon \to 0} \rho(Y + \epsilon X) = \rho(Y). \tag{4.4}$$

It follows from (3.1) that for a linear and diversifying capital allocation Λ_ρ

$$\lim_{\epsilon \to 0-} \Lambda_\rho(X, Y + \epsilon X) \quad \text{and} \quad \lim_{\epsilon \to 0+} \Lambda_\rho(X, Y + \epsilon X)$$

exist. Let U be the linear subspace of V generated by X and Y and define the linear function $f : U \to \mathbb{R}$ by

$$f(\alpha Y + \beta X) := \alpha \cdot \rho(Y) + \beta \cdot \lim_{\epsilon \to 0+} \Lambda_\rho(X, Y + \epsilon X).$$

It follows from (4.4) that for every $\alpha, \beta \in \mathbb{R}$

$$\begin{aligned}
f(\alpha Y + \beta X) &= \alpha \cdot \lim_{\epsilon \to 0+} \Lambda_\rho(Y + \epsilon X, Y + \epsilon X) + \beta \cdot \lim_{\epsilon \to 0+} \Lambda_\rho(X, Y + \epsilon X) \\
&= \lim_{\epsilon \to 0+} \Lambda_\rho(\alpha Y + \beta X, Y + \epsilon X) + \alpha \cdot \lim_{\epsilon \to 0+} \epsilon \Lambda_\rho(X, Y + \epsilon X) \\
&= \lim_{\epsilon \to 0+} \Lambda_\rho(\alpha Y + \beta X, Y + \epsilon X).
\end{aligned}$$

Hence, $f \leq \rho$ on U. By the Hahn-Banach theorem, there exists a linear function h on V with

$$h = f \text{ on } U \quad \text{and} \quad h \leq \rho \text{ on } V.$$

Hence, h is the unique element of H_ρ with $h(Y) = f(Y) = \rho(Y)$ and therefore $h = \Lambda_\rho(., Y)$. It follows that

$$\Lambda_\rho(X, Y) = \lim_{\epsilon \to 0+} \Lambda_\rho(X, Y + \epsilon X).$$

$$\Lambda_\rho(X, Y) = \lim_{\epsilon \to 0-} \Lambda_\rho(X, Y + \epsilon X)$$

is shown in the same way. Hence, Λ_ρ is continuous at Y. $\square$

5 Examples of capital allocation schemes

In this section we consider the most popular risk measures in the finance industry:

1. risk measures based on standard deviations,
2. expected shortfall,
3. value-at-risk.

The objective is to derive explicit allocation formulae for these specific measures.

5.1 Standard deviations

In classical portfolio theory, risk is measured by standard deviations (Markowitz, 1952). Although the significance of this concept is clearly diminished if applied to heavy-tailed distributions, standard deviations are frequently used in the finance industry.

Let p be a positive real number and let L^p denote the set of all random variables $X \in L^0$ such that $|X|^p$ is integrable. In this section we assume that $V := L^2$.

DEFINITION 5.1. Let c be a non-negative real number and define the risk measure ρ_c and the capital allocation Λ_c^{Std} by

$$\begin{array}{lcll} \rho_c(X) & := & c \cdot \mathrm{Std}(X) + \mathrm{E}(X), & \\ \Lambda_c^{Std}(X,Y) & := & c \cdot \mathrm{Cov}(X,Y)/\mathrm{Std}(Y) + \mathrm{E}(X) & \text{if } \mathrm{Std}(Y) > 0, \\ \Lambda_c^{Std}(X,Y) & := & \mathrm{E}(X) & \text{if } \mathrm{Std}(Y) = 0, \end{array}$$

where $\mathrm{E}(X)$ and $\mathrm{Std}(X)$ denote the expectation and the standard deviation of $X \in V$ and $\mathrm{Cov}(X,Y)$ the covariance of $X, Y \in V$.

First we will analyze this parametric class of risk measures.

DEFINITION 5.2. The risk measure ρ is called[2]

$$\begin{array}{lll} \textit{monotonous:} & X \leq Y \ \Rightarrow \ \rho(X) \leq \rho(Y) & \forall X, Y \in V, \\ \textit{translation invariant:} & \rho(X+a) \ = \ \rho(X) + a & \forall a \in \mathbb{R}, \ X \in V. \end{array}$$

The following theorem shows that ρ_c is translation invariant, positively homogeneous and sub-additive. In general, however, it is not monotonous for $c > 0$ (see Fischer (2003) for a class of coherent risk measures based on one-sided moments).

[2]Note that there is a difference in notation compared to Artzner et al. (1999a): in their paper monotonicity is defined by $X \leq Y \ \Rightarrow \ \rho(X) \geq \rho(Y)$. The reason for this difference is that Artzner et al. (1999a) represent portfolios by value functions and not by loss functions.

THEOREM 5.1. *Let c be a non-negative real number.*
(a) *The risk measure $\rho_c(X) = c \cdot \mathrm{Std}(X) + \mathrm{E}(X)$ is translation invariant, positively homogeneous and sub-additive.*
(b) *The risk measure ρ_0 is monotonous. If $c > 0$ and there exists an $A \in \mathcal{A}$ with $0 < \mathbb{P}(A) < c^2/(1+c^2)$ then ρ_c is not monotonous.*

Proof. (a) For every $X, Y \in V$ and $a, b \in \mathbb{R}$,

$$\begin{aligned} \mathrm{E}(aX + bY) &= a \cdot \mathrm{E}(X) + b \cdot \mathrm{E}(Y), \\ \mathrm{Std}(aX + b) &= |a| \cdot \mathrm{Std}(X), \\ \mathrm{Std}(X + Y) &\leq \mathrm{Std}(X) + \mathrm{Std}(Y). \end{aligned}$$

Hence, it immediately follows that $\rho_c(X)$ is translation invariant, positively homogeneous and sub-additive for every $c \geq 0$.

(b) Obviously, $\rho_0(X) = \mathrm{E}(X)$ is monotonous. Let $c > 0$ and $A \in \mathcal{A}$ with $0 < p < c^2/(1+c^2)$, where $p := \mathbb{P}(A)$. Define $X \in V$ with $X = -1/p$ on A and $X = 0$ otherwise. The expectation and variance of X are

$$\begin{aligned} \mathrm{E}(X) &= -1 \\ \mathrm{Var}(X) &= p(-1/p + 1)^2 + (1-p)1^2 = p(1/p - 1)^2 + p(1/p - 1) \\ &= (1/p - 1)(1 - p + p) = 1/p - 1. \end{aligned}$$

Hence,

$$\rho_c(X) = c \cdot \sqrt{1/p - 1} - 1.$$

From $1/p > (1+c^2)/c^2 = (1/c)^2 + 1$ we obtain $\rho_c(X) > 0$ and therefore

$$X \leq 0, \qquad \rho_c(X) > \rho_c(0).$$

Hence, ρ_c is not monotonous. □

By Theorem 4.2, there exists a linear, diversifying capital allocation with respect to ρ_c. The following corollary provides an explicit allocation formula.

Corollary 5.1. *Let c be a non-negative real number. Λ_c^{Std} is a linear, diversifying capital allocation with respect to ρ_c. If $\mathrm{Std}(Y) > 0$ then Λ_c^{Std} is continuous at Y and*

$$\Lambda_c^{Std}(X, Y) = \lim_{\epsilon \longrightarrow 0} \frac{\rho_c(Y + \epsilon X) - \rho_c(Y)}{\epsilon} \tag{5.1}$$

for every $X \in V$.

Proof. Λ_c^{Std} is a linear, diversifying capital allocation with respect to ρ_c because expectations and covariances are linear and $\mathrm{Cov}(X, Y) \leq \mathrm{Std}(X) \cdot \mathrm{Std}(Y)$.

$$\lim_{\epsilon \longrightarrow 0} \mathrm{Cov}(X, Y + \epsilon X) = \mathrm{Cov}(X, Y) \quad \text{and} \quad \lim_{\epsilon \longrightarrow 0} \mathrm{Std}(Y + \epsilon X) = \mathrm{Std}(Y)$$

immediately imply that Λ_c^{Std} is continuous at every $Y \in V$ with $\mathrm{Std}(Y) > 0$. Equality (5.1) follows from Theorem 3.1. $\square$

The fact that ρ_c is not monotonous has unpleasant practical consequences for the covariance allocation Λ_c^{Std}. Let $X, Y \in V$ and assume that $X \leq r$ for a constant $r \in \mathbb{R}$, i.e. the potential losses of X are bounded by r. However, since ρ_c is not monotonous the contributory capital $\Lambda_c^{Std}(X, Y)$ of X in Y might exceed r. In fact, this problem frequently occurs in credit portfolios: if covariance allocation Λ_c^{Std} is used the contributory capital of a loan might be higher than its exposure (Kalkbrener et al., 2004).

5.2 Expected shortfall

The coherency axioms in Artzner et al. (1997, 1999a) provide an excellent framework for the theoretical analysis of risk measures. In particular, this axiomatization highlights weaknesses of traditional measures based on standard deviations or quantiles. As a consequence coherent risk measures are increasingly applied in the finance industry.

Let L^∞ denote the set of all (almost surely) bounded random variables on Ω, i.e. L^∞ consists of all random variables X for which there exists a constant $r \in \mathbb{R}$ with $\mathbb{P}(|X| > r) = 0$. In this subsection we assume that V equals L^∞.

DEFINITION 5.3. A risk measure is called *coherent* if it is monotonous, translation invariant, positively homogeneous and sub-additive.

Let ρ be a coherent risk measure. It follows from Theorem 4.2 that Λ_ρ in (4.2) defines a linear and diversifying capital allocation. For coherent risk measures the real-valued function $\Lambda_\rho(., Y)$ is not only linear and continuous but it can be represented as an expectation with respect to a finitely additive probability (Theorem 2.3 in Delbaen, 2002). If ρ satisfies an additional monotonicity condition (Definition 4 in Delbaen, 2000) then only expectations with respect to σ-additive probability measures have to be considered (Theorems 7 and 8 in Delbaen, 2000).

The most popular class of coherent risk measures is expected shortfall (see, for instance, Rockafellar and Uryasev, 2000, 2001; Acerbi and Tasche, 2002). Let $Y \in V$, $\alpha \in (0, 1)$ and denote the smallest α-quantile by

$$q_\alpha(Y) := \inf\{x \in \mathbb{R} \mid \mathbb{P}(Y \leq x) \geq \alpha\}.$$

The expected shortfall of Y at level α, denoted by ES_α, is the risk measure defined by

$$\mathrm{ES}_\alpha(Y) := (1 - \alpha)^{-1} \int_\alpha^1 q_u(Y) du.$$

It is easy to show that

$$\mathrm{ES}_\alpha(Y) = (1-\alpha)^{-1}(\mathrm{E}(Y\,\mathbf{1}_{\{Y>q_\alpha(Y)\}}) + q_\alpha(Y)\cdot(\mathbb{P}(Y \le q_\alpha(Y)) - \alpha)) \tag{5.2}$$

is an equivalent characterization of expected shortfall. Furthermore, ES_α is coherent (Acerbi and Tasche, 2002) and satisfies the monotonicity condition in Delbaen (2000). Hence, there exists a set $\mathcal{Q}$ of probability measures with

$$\mathrm{ES}_\alpha(Y) = \max\{\mathrm{E}_{\mathbb{Q}}(Y) \mid \mathbb{Q} \in \mathcal{Q}\}.$$

We now construct $\mathcal{Q}$ (see Delbaen, 2000): for every $Y \in V$ let the probability measure $\mathbb{Q}_Y$ be defined by

$$\frac{d\mathbb{Q}_Y}{d\mathbb{P}} := \frac{\mathbf{1}_{\{Y>q_\alpha(Y)\}} + \beta_Y \mathbf{1}_{\{Y=q_\alpha(Y)\}}}{1-\alpha}$$

where

$$\beta_Y := \frac{\mathbb{P}(Y \le q_\alpha(Y)) - \alpha}{\mathbb{P}(Y = q_\alpha(Y))} \quad \text{if} \quad \mathbb{P}(Y = q_\alpha(Y)) > 0.$$

By (5.2),

$$\mathrm{ES}_\alpha(Y) = \mathrm{E}_{\mathbb{Q}_Y}(Y) = \max\{\mathrm{E}_{\mathbb{Q}}(Y) \mid \mathbb{Q} \in \mathcal{Q}\},$$

where $\mathcal{Q}$ is defined by

$$\mathcal{Q} := \{\mathbb{Q}_Y \mid Y \in V\}.$$

According to (4.2),

$$\Lambda_\alpha^{ES}(X,Y) := \mathrm{E}_{\mathbb{Q}_Y}(X) = (\int X \cdot \mathbf{1}_{\{Y>q_\alpha(Y)\}} d\mathbb{P} + \beta_Y \int X \cdot \mathbf{1}_{\{Y=q_\alpha(Y)\}} d\mathbb{P})/(1-\alpha)$$

is a linear, diversifying capital allocation with respect to ES_α (for expected shortfall allocation, see also Schmock and Straumann (1999)). If

$$\mathbb{P}(Y > q_\alpha(Y)) = 1-\alpha \quad \text{or} \quad \mathbb{P}(Y \ge q_\alpha(Y)) = 1-\alpha \tag{5.3}$$

then

$$\lim_{\epsilon \to 0} \frac{d\mathbb{Q}_{Y+\epsilon X}}{d\mathbb{P}} = \frac{d\mathbb{Q}_Y}{d\mathbb{P}} \text{ a.s.}$$

for every $X \in V$ and therefore Λ_α^{ES} is continuous at Y. In particular, (5.3) is satisfied if $\mathbb{P}(Y = q_\alpha(Y)) = 0$.

5.3 Value-at-risk

The value-at-risk $\mathrm{VaR}_\alpha(Y)$ of a portfolio $Y \in V \subseteq L^0$ at level $\alpha \in (0,1)$ is defined as an α-quantile of Y (RiskMetrics, 1995; see Jorion (1997), Duffie and Pan (1997) and Dowd (1998) for an overview). More precisely, in this paper $\mathrm{VaR}_\alpha(Y)$ denotes the smallest α-quantile, i.e. $\mathrm{VaR}_\alpha(Y) = q_\alpha(Y)$. For general portfolios, VaR is not sub-additive and therefore diversification, which is commonly considered as a way to reduce risk, might increase value-at-risk. Despite this shortcoming value-at-risk has become the dominant concept for risk measurement in the finance industry and has even achieved the high status of being written into industry regulations. The development of sound capital allocation techniques for value-at-risk is therefore an important practical problem (Garman, 1996, 1997).

The missing sub-additivity property of value-at-risk prevents the direct application of the axiomatic approach developed in this paper: it follows from Theorem 4.2 that there does not exist a linear, diversifying capital allocation with respect to VaR. However, the results in this paper provide some guidance which techniques to use. In the following we will discuss allocation techniques based on

1. derivatives,
2. covariances,
3. expected shortfall.

Although there does not exist linear, diversifying capital allocations with respect to VaR, the directional derivative

$$\lim_{\epsilon \to 0} \frac{\mathrm{VaR}_\alpha(Y + \epsilon X) - \mathrm{VaR}_\alpha(Y)}{\epsilon} \tag{5.4}$$

might exist for certain portfolios $X, Y \in V$. Hence, Theorem 3.1 motivates to define the contributory capital $\Lambda_\alpha^{\mathrm{VaR}}(X,Y)$ of X in Y by (5.4). This allocation technique has been suggested by Hallerbach (1999) for market risk applications. It works well in a sufficiently continuous setting (see Tasche (1999), Lemus (1999) and Gouriéroux et al. (2000) for criteria which ensure existence of (5.4)). However, in non-continuous or even discrete models directional derivatives usually do not exist or they are not continuous and highly unstable in α.

An alternative approach is to use the covariance allocation scheme analyzed in subsection 5.1. More precisely, we define for $X, Y \in L^2$ with $\mathrm{E}(Y) \le \mathrm{VaR}_\alpha(Y)$

$$\Lambda_\alpha^{\mathrm{VaR}}(X,Y) := \Lambda_{c(Y)}^{\mathrm{Std}}(X,Y),$$

where $c(Y) \in \mathbb{R}$ satisfies

$$\mathrm{VaR}_\alpha(Y) = c(Y) \cdot \mathrm{Std}(Y) + \mathrm{E}(Y).$$

The covariance allocation of portfolio VaR is particularly popular in credit risk applications although the combination of value-at-risk and covariance allocation causes problems if heavy-tailed distributions are involved: the contributory capital of a sub-portfolio might be higher than its standalone capital, the contributory capital of an individual loan might even be higher than its exposure (see Kalkbrener et al. (2004)

for a detailed comparison of covariance allocation and expected shortfall allocation in credit portfolios).

The second problem can be avoided by allocating portfolio VaR by expected shortfall: if $\mathrm{E}(Y) \leq \mathrm{VaR}_\alpha(Y)$ define the contributory capital of X in Y by

$$\Lambda_\alpha^{\mathrm{VaR}}(X, Y) := \Lambda_{\beta(Y)}^{\mathrm{ES}}(X, Y),$$

where $\beta(Y) \in \mathbb{R}$ satisfies

$$\mathrm{VaR}_\alpha(Y) = \mathrm{ES}_{\beta(Y)}(Y). \tag{5.5}$$

Hence, the contributory capital of the subportfolio X reflects its contribution to the tail $\{Y \geq \beta(Y)\}$ of the portfolio loss distribution which is determined by $\mathrm{VaR}_\alpha(Y)$ according to (5.5). This technique for allocating portfolio VaR has been proposed in a number of papers, for instance in Overbeck (2000) and Bluhm et al. (2002).

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