An axiomatic characterization of capital allocations of coherent risk measures

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Abstract

An axiomatic definition of coherent capital allocations is given. It is shown that coherent capital allocations defined by the proposed axiom system are closely linked to coherent risk measures. More precisely, the associated risk measure of a coherent capital allocation is coherent and, conversely, for every coherent risk measure there exists a coherent capital allocation.

1 Introduction

Allocation techniques for risk capital are a prerequisite for portfolio management and performance measurement in the finance industry. In recent years, theoretical and practical aspects of different allocation schemes have been analyzed in a number of papers; see for instance Tasche (1999, 2002, 2007), Delbaen (2000), Overbeck (2000, 2004), Denault (2001), Hallerbach (2003), Fischer (2003), Urban et al. (2004), Cherny (2007), Cherny and Orlov (2007). A simple axiomatization of capital allocation is given in Kalkbrener (2005): the main axioms are the property that the entire risk capital of a portfolio is linearly allocated to its subportfolios and a diversification property ensuring that the capital allocated to a subportfolio X of a larger portfolio Y never exceeds the risk capital of X considered as a stand-alone portfolio. It is shown that for a given risk measure ρ there exists a capital allocation Λ_ρ which satisfies these axioms if and only if ρ is positively homogeneous and subadditive, i.e. ρ has two of the four properties of coherence. However, the question remains open in Kalkbrener (2005) whether a complete axiomatic characterization can be given for capital allocation schemes of risk measures that satisfy all four coherence axioms. In this short note, we propose the following solution: the two remaining axioms of coherent risk measures, i.e. translation invariance and monotonicity, are translated into properties of capital allocations and added to the axioms of linearity and diversification. Coherent capital allocations are defined as capital allocation schemes that satisfy these four axioms. The terminology is justified by the following result: the associated risk measure of a coherent capital allocation

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is coherent and, conversely, for every coherent risk measure ρ there exists a coherent capital allocation Λ_ρ with associated risk measure ρ.

Section 2 provides a short review of the existing literature on capital allocation. In particular, we summarize the main concepts and results in Kalkbrener (2005), which serve as starting point for the present paper. The axiom system defining a coherent capital allocation is given in Section 3 together with the elementary proof of equivalence between coherence of a capital allocation scheme and coherence of the associated risk measure.

2 On the axiomatization of capital allocation

In recent years, the development of appropriate risk measures has been one of the main topics in quantitative risk management, see Föllmer and Schied (2004) for an excellent exposition. The starting point is the seminal paper Artzner et al. (1999). In this paper, an axiomatic approach to the quantification of risk is presented and a set of four axioms is proposed.

Definition 1 (Coherent risk measures) Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, L^0 the space of all equivalence classes of real valued random variables on Ω and V a subspace of the vector space L^0. We will identify each portfolio X with its loss function, i.e. X is an element of V and $X(\omega)$ specifies the loss of X at a future date in state $\omega \in \Omega$. A risk measure ρ is a function from V to $\mathbb{R}$. It is called **coherent** if it is

monotonic:	$X \leq Y \;\Rightarrow\; \rho(X) \leq \rho(Y)$	$\forall X, Y \in V,$
translation invariant:	$\rho(X + a) \;=\; \rho(X) + a$	$\forall a \in \mathbb{R},\; X \in V,$
positively homogeneous:	$\rho(aX) \;=\; a \cdot \rho(X)$	$\forall a \geq 0,\; X \in V,$
subadditive:	$\rho(X + Y) \;\leq\; \rho(X) + \rho(Y)$	$\forall X, Y \in V.$

The application of risk measures in portfolio management or performance measurement requires the allocation of risk capital either to subportfolios or to business units. More formally, assume that a (not necessarily coherent) risk measure ρ has been fixed and let X be a portfolio that consists of subportfolios $X_1, \ldots, X_m$, i.e. $X = X_1 + \ldots + X_m$. The objective is to distribute the risk capital $k := \rho(X)$ of the portfolio X to its subportfolios, i.e. to compute risk contributions $k_1, \ldots, k_m$ of $X_1, \ldots, X_m$ with $k = k_1 + \ldots + k_m$. This equality is sometimes called full allocation property since all of the portfolio risk capital is allocated. Of course, there are other properties of a capital allocation scheme that are desirable from an economic point of view. Delbaen (2000) applies the game-theoretic notion of fairness, Denault (2001) adds further properties of capital allocations and shows the satisfiability of the resulting axiom system by translating concepts from cooperative game theory into the context of capital allocation with a coherent risk measure.

A different axiomatization is given in Kalkbrener (2005). It is based on the assumption that the capital allocated to subportfolio X_i only depends on X_i and X but not on the decomposition of the remainder $X - X_i = \sum_{j \neq i} X_j$ of the portfolio. Hence, a capital allocation can be considered as a function Λ from $V \times V$ to $\mathbb{R}$. Its interpretation is that $\Lambda(X, Y)$ represents the capital allocated to the portfolio X considered as a subportfolio of portfolio Y.

Definition 2 (Axiomatization of capital allocation) A function $\Lambda\colon V \times V \to \mathbb{R}$ is called a capital allocation with associated risk measure ρ if it satisfies the condition $\Lambda(X,X) = \rho(X)$ for all $X \in V$, i.e. if the capital allocated to X (considered as stand-alone portfolio) is the risk capital $\rho(X)$ of X. The following requirements for Λ are proposed. The capital allocation Λ is called

linear:	$\Lambda(aX + bY, Z) = a\Lambda(X,Z) + b\Lambda(Y,Z)$	$\forall a, b \in \mathbb{R}, X, Y, Z \in V,$
diversifying:	$\Lambda(X,Y) \leq \Lambda(X,X)$	$\forall X, Y \in V,$
continuous at $Y \in V$**:**	$\lim_{\epsilon \to 0} \Lambda(X, Y + \epsilon X) = \Lambda(X,Y)$	$\forall X \in V.$

The first two axioms ensure that the risk capital of the portfolio equals the sum of the (contributory) risk capital of its subportfolios and that the capital allocated to a subportfolio X never exceeds the risk capital of X considered as a stand-alone portfolio. If Λ is continuous at $Y \in V$ then small changes to the portfolio Y only have a limited effect on the risk capital of its subportfolios.

The axiomatic approach in Kalkbrener (2005) provides a close link between risk measures and capital allocation rules, which makes it relatively straightforward to translate properties of risk measures into properties of capital allocations. First, given a capital allocation Λ the associated risk measure ρ is obviously given by the values of Λ on the diagonal, i.e. $\rho(X) = \Lambda(X,X)$. Conversely, for a positively homogeneous and subadditive risk measure ρ a corresponding capital allocation Λ_ρ can be constructed as follows: let V^* be the set of real linear functionals on V and for a given risk measure ρ consider the following subset

$$H_\rho := \{h \in V^* \mid h(X) \leq \rho(X) \text{ for all } X \in V\}.$$

It is an easy consequence of the Hahn-Banach Theorem (see, for instance, Theorem II.3.10 in Dunford and Schwartz (1958)) that for a positively homogeneous and subadditive risk measure ρ we have $\rho(X) = \max\{h(X) \mid h \in H_\rho\}$ for all $X \in V$. Hence, for every $Y \in V$ there exists an $h_Y^\rho \in H_\rho$ with $h_Y^\rho(Y) = \rho(Y)$. This allows to define a capital allocation Λ_ρ by

$$\Lambda_\rho(X,Y) := h_Y^\rho(X). \tag{2.1}$$

The set H_ρ can be interpreted as a collection of (generalized) scenarios: the capital allocated to a subportfolio X of portfolio Y is simply the loss of X under scenario h_Y^ρ.

The following theorem (Theorem 4.2 in Kalkbrener (2005)) states the equivalence between positively homogeneous, subadditive (but not necessarily monotonic or translation invariant) risk measures and linear, diversifying capital allocations.

Theorem 1 *Let* $\rho\colon V \to \mathbb{R}$.

a) *If there exists a linear, diversifying capital allocation* Λ *with associated risk measure* ρ *then* ρ *is positively homogeneous and subadditive.*

b) *If* ρ *is positively homogeneous and subadditive then* Λ_ρ *defined by (2.1) is a linear, diversifying capital allocation with associated risk measure* ρ.

Moreover, if a linear, diversifying capital allocation Λ is continuous at a portfolio $Y \in V$ it is uniquely determined by the directional derivative of its associated risk measure, as the next theorem (Theorem 4.3 in Kalkbrener (2005)) shows.

Theorem 2 *Let ρ be a positively homogeneous and subadditive risk measure and $Y \in V$. Then the following three conditions are equivalent:*

a) *Λ_ρ is continuous at Y, i.e. for all $X \in V$ $\lim_{\epsilon\to 0} \Lambda_\rho(X, Y + \epsilon X) = \Lambda_\rho(X, Y)$.*

b) *The directional derivative*

$$\lim_{\epsilon \to 0} \frac{\rho(Y + \epsilon X) - \rho(Y)}{\epsilon} \tag{2.2}$$

exists for every $X \in V$.

c) *There exists a unique $h \in H_\rho$ with $h(Y) = \rho(Y)$.*

If these conditions are satisfied then $\Lambda_\rho(X, Y)$ equals (2.2) for all $X \in V$, i.e. Λ_ρ is given by the Euler principle[1] *or allocation by the gradient.*

If the equivalent conditions in Theorem 2 are not satisfied for $Y \in V$ then the two-sided directional derivative (2.2) does not exist for all X and the capital allocation $\Lambda_\rho(., Y)$ is not uniquely defined. In a recent paper, Cherny and Orlov (2007) propose two different solutions to this problem: either the replacement of the two-sided directional derivative by a one-sided directional derivative or a modification of the axiomatization in Definition 2 obtained by adjusting the continuity axiom and adding law invariance. They show that for a spectral risk measure there exists a unique capital allocation satisfying the proposed axiom system. Capital allocations for spectral risk measures are also investigated in Overbeck (2004).

The Euler principle is an allocation scheme that has been proposed by several authors, see Tasche (2007) for an overview. Tasche (1999) argues that allocation based on the Euler principle provides the right signals for performance measurement. Another justification for the Euler principle is given in Denault (2001) using the framework of cooperative game theory. He shows that the Euler principle is the only fair allocation principle for a differentiable coherent risk measure. The existence of the directional derivative (2.2) is analyzed in a number of papers, e.g. Gourieroux et al. (2000), Tasche (2002), Fischer (2003) and Carlier (2008). Alternative allocation techniques have been explored in the actuarial sciences, see, for example, Myers and Read (2001) and Dhaene et al. (2003). A comparison of different combinations of risk measures and allocation methods can be found in Urban et al. (2004).

In this paper, we only consider static risk measures and capital allocations, i.e., risk is quantified at a fixed date. The extension to dynamic measures and allocations, which take into account the future evolution of losses, has been extensively studied in recent years. We refer to Cheridito et al. (2006) and Cherny (2006) and the papers cited therein.

Another related problem is the optimal sharing of risk between agents characterized by utility functions or, equivalently, by risk measures. For an extensive list of references we refer to the book by Föllmer and Schied (2004) and the research papers Dana and Scarsini (2005) and Jouini et al. (2007).

[1] Recall Euler's well-known rule that states that if $f : S \to \mathbb{R}$ is positively homogeneous and differentiable at $x \in S \subseteq \mathbb{R}^n$, we have $f(x) = \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(x)$.

3 Capital allocation for coherent risk measures

Theorem 1 states the equivalence between linear, diversifying capital allocations and positively homogeneous, subadditive risk measures. These properties of risk measures form two of the four axioms of coherence. The other two axioms are translation invariance and monotonicity. The objective of this paper is to translate these properties of a risk measure into equivalent properties of the corresponding capital allocation and, as a consequence, to obtain a complete axiomatic characterization of capital allocations of coherent risk measures.

Definition 3 (Coherent capital allocations) A capital allocation $\Lambda : V \times V \to \mathbb{R}$ is called

translation invariant: $\Lambda(X+a, X+b) \;=\; \Lambda(X,X) + a \quad \forall a, b \in \mathbb{R},\ X \in V,$
monotonic: $X \le Y \;\Rightarrow\; \Lambda(X,Z) \le \Lambda(Y,Z) \qquad \forall X, Y, Z \in V.$

A capital allocation $\Lambda : V \times V \to \mathbb{R}$ is called **coherent** if it is linear, diversifying, translation invariant and monotonic.

The following theorem justifies the term coherent capital allocation.

Theorem 3 *Let* $\rho\colon V \to \mathbb{R}$.
(a) *If there exists a coherent capital allocation* Λ *with associated risk measure* ρ *then* ρ *is coherent.*
(b) *If* ρ *is coherent then* Λ_ρ *is a coherent capital allocation with associated risk measure* ρ.

The proof of this theorem is an immediate consequence of Theorem 1 and the following two lemmas.

Lemma 1 *Let* Λ *be a linear and diversifying capital allocation with associated risk measure* ρ. *Then* Λ *is translation invariant if and only if* ρ *is translation invariant.*

Proof: $(\Rightarrow)$ For every $a \in \mathbb{R}$ and $X \in V$

$$\rho(X+a) = \Lambda(X+a, X+a) = \Lambda(X,X) + a = \rho(X) + a.$$

$(\Leftarrow)$ The following sequence of equalities and inequalities is an immediate consequence of the fact that Λ is linear and diversifying and ρ is translation invariant and, by Theorem 1, positively homogeneous:

$$a = \rho(X) - \rho(X-a) \le \Lambda(X,X) - \Lambda(X-a, X) = \Lambda(a, X) \le \Lambda(a,a) = \rho(a) = a$$

for every $a \in \mathbb{R}$ and $X \in V$. Hence, $\Lambda(a, X) = a$. The linearity of Λ, the translation invariance of ρ and $\Lambda(a, X) = a$ imply

$$\rho(X) + a = \rho(X+a) = \Lambda(X+a, X+a) = \Lambda(X, X+a) + \Lambda(a, X+a) = \Lambda(X, X+a) + a$$

and therefore $\Lambda(X, X+a) = \Lambda(X,X)$. Together with $\Lambda(a, X) = a$ and the linearity of Λ,

$$\Lambda(X+a, X+b) = \Lambda(X, X+b) + \Lambda(a, X+b) = \Lambda(X,X) + a$$

for every $a, b \in \mathbb{R}$ and $X \in V$. □

We now turn to the monotonicity property of capital allocations and show its equivalence to the monotonicity of risk measures. A third equivalent formulation is based on the essential supremum of a loss variable $X \in V$.

Lemma 2 *Let V_b be the subset of V consisting of those elements $X \in V$ whose essential supremum*

$$\operatorname{ess\,sup}(X) := \inf\{r \in \mathbb{R} \mid \mathbb{P}(X > r) = 0\}$$

exists in $\mathbb{R}$. Let Λ be a linear, diversifying and translation invariant capital allocation with associated risk measure ρ. Then the following three conditions are equivalent:

a) $\Lambda(X, Y) \le \operatorname{ess\,sup}(X)$ *for every $X \in V_b$ and $Y \in V$.*

b) *The capital allocation Λ is monotonic.*

c) *The risk measure ρ is monotonic.*

Proof: (a) $\Rightarrow$ (b) : Let $X, Y, Z \in V$ with $X \le Y$. Hence, $X - Y \le 0$ and therefore,

$$\Lambda(X - Y, Z) \le \operatorname{ess\,sup}(X - Y) \le 0.$$

The linearity of Λ implies $\Lambda(X, Z) \le \Lambda(Y, Z)$.

(b) $\Rightarrow$ (c) : Let $X, Y \in V$ with $X \le Y$. Since Λ is a diversifying and monotonic capital allocation with associated risk measure ρ we obtain $\rho(X) = \Lambda(X, X) \le \Lambda(Y, X) \le \Lambda(Y, Y) = \rho(Y)$.

(c) $\Rightarrow$ (a) : For every $X \in V_b$ we have $\mathbb{P}(X > \operatorname{ess\,sup}(X)) = 0$ and therefore $X \le \operatorname{ess\,sup}(X)$. Hence, for every $Y \in V$,

$$\Lambda(X, Y) \le \Lambda(X, X) = \rho(X) \le \rho(\operatorname{ess\,sup}(X)), \tag{3.1}$$

because Λ is diversifying and ρ is monotonic. It follows from Theorem 1 and Lemma 1 that ρ is subadditive, positively homogeneous and translation invariant. Hence, $\rho(\operatorname{ess\,sup}(X)) = \operatorname{ess\,sup}(X)$ and therefore, by (3.1), $\Lambda(X, Y) \le \operatorname{ess\,sup}(X)$. □

The essential supremum of X can be interpreted as its highest potential loss. Hence, the risk contribution $\Lambda(X, Y)$ of a portfolio X cannot exceed its highest potential loss if $\Lambda(X, Y) \le \operatorname{ess\,sup}(X)$ holds. In particular, it is ensured that the risk capital allocated to a single loan does not exceed its exposure. This is an essential property for the acceptance of a capital allocation scheme in risk management. Covariance allocation is a frequently used non-monotonic capital allocation scheme without this property. The case study in Kalkbrener et al. (2004) shows that this may cause serious problems if covariance allocation is applied to realistic credit portfolios.

For the rest of the paper let Λ be a linear, diversifying capital allocation with associated risk measure ρ. If ρ is translation invariant then, by Lemma 1, Λ is translation invariant. We will now analyze whether Λ also satisfies the following condition, which is a stronger version of translation invariance:

$$\Lambda(X + a, Y + b) = \Lambda(X, Y) + a \qquad \forall a, b \in \mathbb{R},\ X, Y \in V. \tag{3.2}$$

First, assume that Λ is continuous at $Y \in V$. It easily follows from Theorem 2 that Λ is continuous at $Y + b$ for every $b \in \mathbb{R}$ and

$$\Lambda(X + a, Y + b) = \lim_{\epsilon \to 0} \frac{\rho(Y + \epsilon X) + b + \epsilon a - \rho(Y) - b}{\epsilon} = \Lambda(X, Y) + a$$

for every $X \in V$ and $a, b \in \mathbb{R}$. Hence, (3.2) is satisfied. However, strong translation invariance (3.2) does not necessarily hold if Λ is not continuous at Y.

Example 1 Let Ω be a probability space that consists of two states ω_1 and ω_2 with $\mathbb{P}(\{\omega_1\}) = \mathbb{P}(\{\omega_2\}) = 0.5$. Let $\rho = \mathrm{ES}_\alpha$ be the risk measure expected shortfall at level $\alpha = 0.5$ and $\Lambda = \Lambda_\alpha^{ES}$ the corresponding capital allocation, i.e. for $X, Y \in V$ the expected shortfall allocation Λ is defined by

$$\begin{aligned} \Lambda(X,Y) &:= X(\omega_1) \quad \text{if} \quad Y(\omega_1) > Y(\omega_2), \\ \Lambda(X,Y) &:= X(\omega_2) \quad \text{if} \quad Y(\omega_2) > Y(\omega_1), \\ \Lambda(X,Y) &:= (X(\omega_1) + X(\omega_2))/2 \quad \text{if} \quad Y(\omega_1) = Y(\omega_2). \end{aligned}$$

The capital allocation Λ is linear, diversifying and satisfies the strong translation invariance (3.2). Furthermore, Λ is continuous at Y if and only if $Y(\omega_1) \neq Y(\omega_2)$.

We will now define another capital allocation $\tilde{\Lambda}$ with associated risk measure $\rho = \mathrm{ES}_\alpha$ that differs from Λ only at $Y = 0$. More precisely, define for $X, Y \in V$

$$\tilde{\Lambda}(X,Y) := \Lambda(X,Y) \ \text{ if } Y \neq 0, \qquad \tilde{\Lambda}(X,0) := X(\omega_1).$$

It is easy to see that $\tilde{\Lambda}$ is linear, diversifying and translation invariant. However, $\tilde{\Lambda}$ does not satisfy the stronger property (3.2): for $X, Y \in V$ with $X(\omega_1) = 1$, $X(\omega_2) = 0$ and $Y(\omega_1) = Y(\omega_2) = 0$ and real numbers $a = 0$ and $b = 1$ we have

$$\tilde{\Lambda}(X+a, Y+b) = \tilde{\Lambda}(X,1) = 1/2 \neq 1 = \tilde{\Lambda}(X,0) = \tilde{\Lambda}(X,Y) + a.$$

In summary, if a capital allocation Λ is linear and diversifying and the associated risk measure ρ is translation invariant then Λ is always translation invariant but does not necessarily satisfy (3.2). However, we will now show that for a given translation invariant risk measure ρ there always exists a capital allocation Λ_ρ with this property. The construction of Λ_ρ is achieved by putting an additional constraint on Λ_ρ in definition (2.1). More precisely, we define an equivalence relation $\sim$ on V: two elements $X, Y \in V$ are equivalent if they differ by a constant, i.e. $X \sim Y$ if there exists an $a \in \mathbb{R}$ with $Y = X + a$. For every equivalence class $Y_\sim$ we choose an arbitrary element $\tilde{Y} \in Y_\sim$ and define the capital allocation Λ_ρ by

$$\Lambda_\rho(X,Y) := h^\rho_{\tilde{Y}}(X), \tag{3.3}$$

where $h^\rho_{\tilde{Y}}$ is an arbitrary element of H_ρ with $h^\rho_{\tilde{Y}}(\tilde{Y}) = \rho(\tilde{Y})$. Note that the same scenario $h^\rho_{\tilde{Y}_1}(\cdot) = h^\rho_{\tilde{Y}_2}(\cdot)$ is used for portfolios Y_1, Y_2 that differ only by a constant.

For a risk measure ρ that is not only positively homogeneous and subadditive but also translation invariant the capital allocation defined by (3.3) can be considered as a refinement of (2.1). More precisely, for every $Y \in V$ the generalized scenario $h^\rho_{\tilde{Y}}(\cdot)$ used in the definition (3.3) of $\Lambda_\rho(X,Y)$ satisfies the condition

$$h^\rho_{\tilde{Y}}(Y) = \rho(Y) \tag{3.4}$$

in definition (2.1). The proof of (3.4) is based on the observation that $a \cdot h(1) = h(a) \leq \rho(a) = a$ for every $h \in H_\rho$ and $a \in \mathbb{R}$. Therefore, $h(1) = 1$ and, more general, $h(a) = a$. For every $Y \in V$

there exists a $b \in \mathbb{R}$ with $Y = \tilde{Y} + b$. The equality $h^\rho_{\tilde{Y}}(b) = b$ together with the linearity of $h^\rho_{\tilde{Y}}$ and the translation invariance of ρ imply (3.4):

$$h^\rho_{\tilde{Y}}(Y) = h^\rho_{\tilde{Y}}(Y-b) + b = h^\rho_{\tilde{Y}}(\tilde{Y}) + b = \rho(\tilde{Y}) + b = \rho(\tilde{Y}+b) = \rho(Y).$$

It immediately follows from (3.4), Theorem 1 and Lemma 1 that Λ_ρ defined in (3.3) is a linear, diversifying and translation invariant capital allocation with associated risk measure ρ. Together with the definition and linearity of Λ_ρ we obtain

$$\Lambda_\rho(X+a, Y+b) = \Lambda_\rho(X+a, Y) = \Lambda_\rho(X,Y) + \Lambda_\rho(Y,Y) - \Lambda_\rho(Y-a, Y) = \Lambda_\rho(X,Y) + a$$

for every $X, Y \in V$ and $a, b \in \mathbb{R}$. Hence, we have shown the following result.

Lemma 3 *Let $\rho\colon V \to \mathbb{R}$ be a positively homogeneous, subadditive and translation invariant risk measure. Then Λ_ρ defined by (3.3) is a linear, diversifying capital allocation with associated risk measure ρ that satisfies the property of strong translation invariance (3.2).*

As a consequence, Theorem 3 remains correct if translation invariance is replaced by the stronger property (3.2) in the definition of coherent capital allocations.

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