Implicitization of Rational Parametric Curves and Surfaces

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Abstract

In this paper we use Gröbner bases for the implicitization of rational parametric curves and surfaces in 3D-space. We prove that the implicit form of a curve or surface given by the rational parametrization

$$x_1 := \frac{p_1}{q_1} \qquad x_2 := \frac{p_2}{q_2} \qquad x_3 := \frac{p_3}{q_3},$$

where the p's and q's are univariate polynomials in y_1 or bivariate polynomials in y_1, y_2 over a field K, can always be found by computing

$$GB(\{q_1 \cdot x_1 - p_1,\ q_2 \cdot x_2 - p_2,\ q_3 \cdot x_3 - p_3\}) \cap K[x_1, x_2, x_3],$$

where GB is the Gröbner basis with respect to the lexical ordering with $x_1 \prec x_2 \prec x_3 \prec y_1 \prec y_2$, if for every $i, j \in \{1, 2, 3\}$ with $i \neq j$ the polynomials p_i, q_i, p_j, q_j have no common zeros. This result leads immediately to an implicitization algorithm for arbitrary rational parametric curves.

Furthermore, we present an algorithm for the implicitization of arbitrary rational parametric surfaces and prove its termination and correctness.

1 Introduction

The automatic conversion of parametrically defined varieties into their implicit form is of fundamental importance in geometric modeling. The reason for this is that implicit and parametric representations are appropriate for different classes of problems. For instance, it is universally recognized that the parametric representation is best suited for generating points along a variety, whereas the implicit representation is most convenient for determining whether a given point lies on a specific variety. It is also well-known that the problem of intersecting two varieties is greatly simplified if one variety can be expressed implicitly and the other parametrically.

For some time the implicitization problem has been deemed unsolvable in the CAD literature ([4] or [11]). In 1984 the problem has been solved for rational parametric curves in 2D and rational parametric surfaces in 3D by using resultants (see [10]). Resolvents have been applied to find the implicit representation of rational parametric cubic curves in 3D ([5]). Recently, algorithms based on resultants have been developed for solving the implicitization problem for rational parametric surfaces ([3] and [9]). Arnon and Sederberg used Gröbner bases for the implicitization of polynomial parametric varieties of dimension $n-1$ in n-dimensional space ([1]). In 1987 Buchberger generalized their method to the case of polynomial parametric varieties of arbitrary dimension ([2]). Recently, we applied Gröbner bases to the most general problem, the implicitization of rational parametric varieties of arbitrary dimension in arbitrary dimensional space ([8]). Many of the implicitization methods are highlighted in [7].

In this paper we use Gröbner bases for the implicitization of rational parametric curves and surfaces in 3D-space. In contrast to the algorithms for the implicitization of m-dimensional varieties in n-dimensional space presented in [8] the algorithms in this paper work without introducing new variables. Therefore they solve the implicitization problem in 3D-space much faster than the general algorithms. (A comparision of the computing times of our implementations in Maple can be found in [8]).

In this paper we prove that the implicit form of a curve or surface given by the rational parametrization

$$x_1 := \frac{p_1}{q_1} \qquad x_2 := \frac{p_2}{q_2} \qquad x_3 := \frac{p_3}{q_3},$$

where the p's and q's are univariate polynomials in y_1 or bivariate polynomials in y_1, y_2 over a field K, can always be found by computing

$$GB(\{q_1 \cdot x_1 - p_1,\ q_2 \cdot x_2 - p_2,\ q_3 \cdot x_3 - p_3\}) \cap K[x_1, x_2, x_3],$$

where GB is the Gröbner basis with respect to the lexical ordering with $x_1 \prec x_2 \prec x_3 \prec y_1 \prec y_2$, if for every $i, j \in \{1, 2, 3\}$ with $i \neq j$ the polynomials p_i, q_i, p_j, q_j have no common zeros. Since we can always assume that p_i and q_i are relatively prime ($i = 1, 2, 3$), the above condition is always satisfied, if the p's and q's are univariate. Therefore, the above result leads immediately to an implicitization algorithm for arbitrary rational parametric curves.

Furthermore, we present an algorithm for the implicitization of arbitrary rational parametric surfaces and prove its termination and correctness.

In section 2 we state the problems we are concerned with. In section 3 a few theorems are proved which are necessary for showing the correctness of the algorithms, which we present in section 4.

2 Problems

Throughout the paper let K be a field and $\bar{K}$ the algebraic closure of K.
Let J be an ideal and $g_1, \dots, g_m$ polynomials in $K[x_1, \dots, x_n]$. $V(J)$ denotes the *variety* of J, i.e. the set

$$\{a \in \bar{K}^n \mid f(a) = 0 \text{ for every } f \in J\}.$$

Instead of $V(Ideal(\{g_1, \dots, g_m\}))$ we will often write $V(\{g_1, \dots, g_m\})$.
Let L be a field with $K \subseteq L$. Then $(a_1, \dots, a_n) \in L^n$ is a *generic point* of J if for every $f \in K[x_1, \dots, x_n]$:

$$f \in J \quad \text{iff} \quad f(a_1, \dots, a_n) = 0.$$

It is well-know that an ideal is prime if and only if it has a generic point with coordinates in a universal domain (see for instance [12]).

In this paper we want to solve the following two problems:

Implicitization Problem for Rational Parametric Curves:
given: rational parametrization of a curve

$$x_1 = \frac{p_1}{q_1} \qquad x_2 = \frac{p_2}{q_2} \qquad x_3 = \frac{p_3}{q_3},$$

where $p_1, p_2, p_3 \in K[y_1]$, $q_1, q_2, q_3 \in K[y_1] - \{0\}$ and p_i and q_i are relatively prime ($i = 1, 2, 3$).
find: implicit representation of this curve, i.e. polynomials $g_1, \dots, g_m$ in $K[x_1, x_2, x_3]$ such that

$$V(\{g_1, \dots, g_m\}) = V(P'),$$

where P' is the prime ideal in $K[x_1, x_2, x_3]$ with

$$(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}) \in K(y_1)^3$$

as generic point.

Implicitization Problem for Rational Parametric Surfaces:
given: rational parametrization

$$x_1 = \frac{p_1}{q_1} \qquad x_2 = \frac{p_2}{q_2} \qquad x_3 = \frac{p_3}{q_3},$$

where $p_1, p_2, p_3 \in K[y_1, y_2]$, $q_1, q_2, q_3 \in K[y_1, y_2] - \{0\}$ and p_i and q_i are relatively prime ($i = 1, 2, 3$).
decide: whether the parametric object is a surface, i.e. whether the transcendence degree of

$$K(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3})$$

(over K) is 2. In this case
find: implicit representation of this surface, i.e. a polynomial g in $K[x_1, x_2, x_3]$ such that

$$V(\{g\}) = V(P'),$$

where P' is the prime ideal in $K[x_1, x_2, x_3]$ with

$$(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}) \in K(y_1, y_2)^3$$

as generic point.

Example 1 *For the rational parametrization*

$$x_1 = \frac{2y_2}{1 + y_1^2 + y_2^2} \qquad x_2 = \frac{2y_1y_2}{1 + y_1^2 + y_2^2} \qquad x_3 = \frac{y_2^2 - y_1^2 - 1}{1 + y_1^2 + y_2^2}$$

the implicit representation

$$x_1^2 + x_2^2 + x_3^2 - 1$$

of the unit sphere is a solution of the above problem. □

3 Theorems

Throughout the paper let $p_1, p_2, p_3 \in K[y_1, y_2]$ and $q_1, q_2, q_3 \in K[y_1, y_2] - \{0\}$ such that p_i and q_i are relatively prime ($i = 1, 2, 3$). Let

$$f_1 := q_1 \cdot x_1 - p_1, \quad f_2 := q_2 \cdot x_2 - p_2, \quad f_3 := q_3 \cdot x_3 - p_3,$$

$$I := Ideal(\{f_1, f_2, f_3\}) \text{ in } K[x_1, x_2, x_3, y_1, y_2]$$

and let $Q_1, \ldots, Q_r$ be primary ideals in $K[x_1, x_2, x_3, y_1, y_2]$ such that $Q_1 \cap \ldots \cap Q_r$ is a reduced primary decomposition of I. Furthermore, P denotes the prime ideal in the polynomial ring $K[x_1, x_2, x_3, y_1, y_2]$ which has

$$(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}, y_1, y_2) \in K(y_1, y_2)^5$$

as generic point.

Theorem 1 *There exists an* $i \in \{1, \ldots, r\}$ *with*

$$Q_i = P$$

and for every $j \in \{1, \ldots, r\} - \{i\}$*:*

$$Q_j \cap K[y_1, y_2] \neq \{0\}.$$

Proof: In this proof we use the following notation:
For a given ideal F in $K[x_1, x_2, x_3, y_1, y_2]$ the ideal in $K(y_1, y_2)[x_1, x_2, x_3]$ generated by F is denoted by $F^\star$.

Obviously, $I^\star$ is a zero-dimensional prime ideal. By [6] p.92, there exists exactly one element i of $\{1, \ldots, r\}$ with

$$Q_i \cap K[y_1, y_2] = \{0\}.$$

Furthermore, $I^\star = Q_i^\star$. By [6] p.47, $P^\star$ is a zero-dimensional prime ideal. As $I \subseteq P$,

$$P^\star = I^\star = Q_i^\star.$$

Using [6] p.92 again,

$$Q_i = P. \quad \square$$

For the rest of the paper let us assume that

$$Q_1 = P$$

and that $Q_2, \ldots, Q_r$ are ordered in such a way that there exists a $v \in \{1, \ldots, r\}$ such that

$Q_1, \ldots, Q_v$ are isolated primary components and

$Q_{v+1}, \ldots, Q_r$ are embedded primary components.

Obviously,

$$V(I) = V(P_1) \cup \ldots \cup V(P_v), \tag{1}$$

where P_i is the radical of Q_i for $i = 1, \ldots, r$.
By Krull's Primidealkettensatz (see for instance [6] p.179),

$$dim(P_j) \geq 2 \qquad (j = 1, \ldots, v), \tag{2}$$

where $dim(P_j)$ denotes the dimension of P_j.

Definition: Let $(b_1, b_2) \in \bar{K}^2$. We denote the number of elements in the set

$$\{i \in \{1, 2, 3\} \mid p_i(b_1, b_2) = q_i(b_1, b_2) = 0\}$$

by $zero(b_1, b_2)$.

Example 2 *We consider again the parametrization*

$$x_1 = \frac{2y_2}{1 + y_1^2 + y_2^2} \qquad x_2 = \frac{2y_1y_2}{1 + y_1^2 + y_2^2} \qquad x_3 = \frac{y_2^2 - y_1^2 - 1}{1 + y_1^2 + y_2^2}$$

of the unit sphere. Then for $(0, 0)$, $(i, 0) \in \bar{Q}^2$, *where* Q *denotes the field of rational numbers:*

$$zero(0, 0) = 0 \quad \textit{and} \quad zero(i, 0) = 3. \quad \square$$

Theorem 2 *Let $j \in \{2, \dots, v\}$ and $(a_1, a_2, a_3, b_1, b_2)$ the generic point of the prime ideal P_j in $K[x_1, x_2, x_3, y_1, y_2]$. Then*

$$b_1, b_2 \in \bar{K} \text{ and } dim(P_j) \leq zero(b_1, b_2).$$

Proof: First of all, we know from Theorem 1 that the transcendence degree of $K(b_1, b_2)$ is smaller than 2.
Let us assume that the transcendence degree of $K(b_1, b_2)$ is 1.
Let $i \in \{1, 2, 3\}$. From the fact that p_i, q_i are relatively prime it follows that (b_1, b_2) is no common zero of p_i and q_i. As f_i is an element of P_j, a_i is algebraically dependent on $\{b_1, b_2\}$. Thus, $dim(P_j) = 1$. This is a contradiction to (2).
Therefore,

$$b_1, b_2 \in \bar{K}.$$

If (b_1, b_2) is no common zero of p_i and q_i then a_i is algebraically dependent on $\{b_1, b_2\}$. Thus, the transcendence degree of $K(a_1, a_2, a_3, b_1, b_2)$ is less equal $zero(b_1, b_2)$. Therefore,

$$dim(P_j) \leq zero(b_1, b_2). \quad \Box$$

Theorem 3

$$V(I) \neq V(P)$$

implies

that there exists a $(b_1, b_2) \in \bar{K}^2$ with $zero(b_1, b_2) \geq 2$.

Proof: If $V(I) \neq V(P)$ then we obtain from (1) that v is greater equal 2.
Let $(a_1, a_2, a_3, b_1, b_2)$ be the generic point of P_2. By Theorem 2 and (2),

$$(b_1, b_2) \in \bar{K}^2 \text{ and } zero(b_1, b_2) \geq 2. \quad \Box$$

4 Algorithms

If for every $(i, j) \in \{(1, 2), (1, 3), (2, 3)\}$

$$p_i, q_i, p_j, q_j \text{ have no common zeros}$$

then, by Theorem 3,

$$V(I \cap K[x_1, x_2, x_3]) = V(P \cap K[x_1, x_2, x_3]).$$

In this case it follows from the elimination property of Gröbner bases that we can obtain the implicit form of the curve or the surface given by

$$x_1 = \frac{p_1}{q_1} \qquad x_2 = \frac{p_2}{q_2} \qquad x_3 = \frac{p_3}{q_3}$$

by computing

$$\{g_1, \ldots, g_m\} := GB(\{q_1 \cdot x_1 - p_1,\ q_2 \cdot x_2 - p_2,\ q_3 \cdot x_3 - p_3\}) \cap K[x_1, x_2, x_3],$$

where GB has to be computed using the lexical ordering determined by $x_1 \prec x_2 \prec x_3 \prec y_1 \prec y_2$.

In particular, if a polynomial parametric surface or a rational parametric curve is given we obtain from Theorem 3:

Corollary 1
a) **(Parametrization by polynomial functions:)**
If $q_1 = q_2 = q_3 = 1$ *then* $V(I) = V(P)$.
b) **(Rational parametrization of curves:)**
If $p_1, p_2, p_3, q_1, q_2, q_3 \in K[y_1]$ *then* $V(I) = V(P)$.

Hence, the simple algorithm described above solves the implicitization problem for rational parametric curves.

It is an easy consequence of Theorem 2 that

$$I \cap K[x_1, x_2, x_3] = \{0\}$$

iff

there exists a $(b_1, b_2) \in \bar{K}^2$ with $zero(b_1, b_2) = 3$.

Therefore, if there exists such a (b_1, b_2) then every technique from elimination theory must fail in finding an implicit representation.

Example 3 *The implicit equation of the unit sphere cannot be found by computing*

$$GB(\{q_1 \cdot x_1 - p_1,\ q_2 \cdot x_2 - p_2,\ q_3 \cdot x_3 - p_3\}) \cap K[x_1, x_2, x_3],$$

where the p's and q's are defined as in Example 1 or 2:
Since there exists a $(b_1, b_2) \in \bar{K}^2$ *with* $zero(b_1, b_2) = 3$ *(see Example 2),*

$$Ideal(\{q_1 \cdot x_1 - p_1,\ q_2 \cdot x_2 - p_2,\ q_3 \cdot x_3 - p_3\}) \cap K[x_1, x_2, x_3] = \{0\}$$

and therefore

$$GB(\{q_1 \cdot x_1 - p_1,\ q_2 \cdot x_2 - p_2,\ q_3 \cdot x_3 - p_3\}) \cap K[x_1, x_2, x_3] = \emptyset. \quad \square$$

The same problem is addressed in [3] and [9]. In these papers parametrizations of that kind are called parametrizations with base points. Resultant techniques are used to compute implicit representations.

In this paper we use Gröbner bases for solving the implicitization problem for rational parametric surfaces.

Definition: Let h, g be polynomials in $K[x_1, x_2, x_3, y_1]$ such that g has no non-trivial factor in $K[y_1]$ and there exists a polynomial p in $K[y_1]$ with $h = g \cdot p$. Then

$$h_{/y_1} := g.$$

implicit_surface (in: $p_1, p_2, p_3, q_1, q_2, q_3$; out: g)

input: $p_1, p_2, p_3 \in K[y_1, y_2]$, $q_1, q_2, q_3 \in K[y_1, y_2] - \{0\}$ and

$$p_i \text{ and } q_i \text{ are relatively prime} \quad (i = 1, 2, 3).$$

output: $g \in K[x_1, x_2, x_3]$ such that if the transcendence degree of

$$K(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3})$$

is 2 then

$$g \notin K \text{ and } V(\{g\}) = V(P'),$$

where P' is the prime ideal in $K[x_1, x_2, x_3]$ with the generic point

$$(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}),$$

and

$$g = 1$$

otherwise.

for every $(i, j) \in \{(1, 2), (1, 3), (2, 3)\}$ **do**
    $G_{(i,j)} := GB(\{f_i, f_j\}) \cap K[x_1, x_2, x_3, y_1]$, where $f_k := q_k \cdot x_k - p_k \quad (k = 1, 2, 3)$
    $F_{(i,j)} := \{h_{/y_1} \mid h \in G_{(i,j)}\}$
$G := GB(F_{(1,2)} \cup F_{(1,3)} \cup F_{(2,3)} \cup \{f_1, f_2, f_3\}) \cap K[x_1, x_2, x_3]$
$g := gcd(G)$

where GB has to be computed using the lexical ordering determined by $x_1 \prec x_2 \prec x_3 \prec y_1 \prec y_2$.

Example 4 *Again we consider the unit sphere given by*

$$x_1 = \frac{2y_2}{1+y_1^2+y_2^2} \qquad x_2 = \frac{2y_1y_2}{1+y_1^2+y_2^2} \qquad x_3 = \frac{y_2^2-y_1^2-1}{1+y_1^2+y_2^2}.$$

Using **implicit_surface** *we obtain*

$G_{(1,2)} := \{x_2 + y_1^2x_2 - x_1y_1 - y_1^3x_1\}$,
$F_{(1,2)} := \{-x_2 + x_1y_1\}$,
$G_{(1,3)} := \{x_1^2 + 2x_1^2y_1^2 - y_1^2 - 1 + y_1^4x_1^2 + x_3^2 + y_1^2x_3^2\}$,
$F_{(1,3)} := \{x_1^2y_1^2 + x_1^2 - 1 + x_3^2\}$,
$G_{(2,3)} := \{-x_2^2 - 2y_1^2x_2^2 + y_1^4 + y_1^2 - y_1^4x_2^2 - y_1^2x_3^2 - y_1^4x_3^2\}$,
$F_{(2,3)} := \{y_1^2x_2^2 + x_2^2 - y_1^2 + y_1^2x_3^2\}$,
$G := \{x_1^2 + x_2^2 + x_3^2 - 1\}$,
$g := x_1^2 + x_2^2 + x_3^2 - 1$, *the implicit representation of the unit sphere.* □

As termination of the algorithm is obvious it remains to prove its correctness.

Proof of correctness:
Let

$$\bar{I} := Ideal(F_{(1,2)} \cup F_{(1,3)} \cup F_{(2,3)} \cup \{f_1, f_2, f_3\}),$$

$\bar{P}$ a prime ideal in $K[x_1, x_2, x_3, y_1, y_2]$ with $\bar{I} \subseteq \bar{P}$ and $P \neq \bar{P}$ and let $(a_1, a_2, a_3, b_1, b_2)$ be the generic point of $\bar{P}$.

Assumption: $dim(\bar{P}) > 1$.
Then,

$$P \not\subseteq \bar{P}.$$

As $I \subseteq \bar{I} \subseteq \bar{P}$ there exists an $i \in \{2, \ldots, v\}$ with $P_i \subseteq \bar{P}$. By Theorem 2,

$$b_1, b_2 \in \bar{K}.$$

As $dim(\bar{P}) > 1$ there exist $j, k \in \{1, 2, 3\}$ such that $j \neq k$ and $\{a_j, a_k\}$ is algebraically independent over K. Since p_j and q_j are relatively prime and $gcd(f_j, f_k)$ divides p_j and q_j,

$$gcd(f_j, f_k) = 1.$$

Thus, $Ideal(\{f_j, f_k\}) \cap K[x_1, x_2, x_3, y_1] \neq \{0\}$ and therefore there exists a non-zero polynomial $f(x_j, x_k, y_1) \in F_{(j,k)}$. By definition of $F_{(j,k)}$,

$$f(x_j, x_k, b_1) \neq 0.$$

This is a contradiction to the fact that $\{a_j, a_k\}$ is algebraically independent over K.

Thus, P is the only prime ideal that is a superideal of $\bar{I}$ and has a dimension greater than 1. Hence, $\bar{I}$ can be written in the form

$$P \cap R,$$

where R is an ideal in $K[x_1, x_2, x_3, y_1, y_2]$ with $dim(R) < 2$. Therefore,

$$\bar{I} \cap K[x_1, x_2, x_3] = (P \cap K[x_1, x_2, x_3]) \cap (R \cap K[x_1, x_2, x_3]) \text{ and } dim(R \cap K[x_1, x_2, x_3]) < 2. \quad (3)$$

It follows from the elimination property of Gröbner bases that

$$G \text{ is a basis of } \bar{I} \cap K[x_1, x_2, x_3].$$

Case:

$$\text{the transcendence degree of } K(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}) \text{ is 2.}$$

In this case $P \cap K[x_1, x_2, x_3]$ is a prime ideal of dimension 2. Thus, there exists an $h \in K[x_1, x_2, x_3] - K$ with $Ideal(\{h\}) = P \cap K[x_1, x_2, x_3]$. As $G \subseteq P \cap K[x_1, x_2, x_3]$,

$$h \text{ divides } gcd(G).$$

Let $p \in K[x_1, x_2, x_3]$ such that $gcd(G) = h \cdot p$. Obviously, p divides every polynomial in $R \cap K[x_1, x_2, x_3]$. As the dimension of $R \cap K[x_1, x_2, x_3]$ is less than 2, p is a non-zero constant. Thus,

$$V(\{gcd(G)\}) = V(\{h\}) = V(P \cap K[x_1, x_2, x_3]) = V(P').$$

Case:

$$\text{the transcendence degree of } K(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}) \text{ is less than 2.}$$

In this case $dim(P \cap K[x_1, x_2, x_3])$ is less than 2 and therefore, by (3), $dim(\bar{I} \cap K[x_1, x_2, x_3])$ is less than 2. Hence,

$$gcd(G) = 1. \quad \square$$

Some of the Gröbner bases computations in **implicit_surface** can be replaced by other elimination methods, for instance by computations of Sylvester resultants:
We can replace

$G_{(i,j)} := GB(\{f_i, f_j\}) \cap K[x_1, x_2, x_3, y_1]$

by

$G_{(i,j)} := \{resultant(f_i, f_j)\}$, where f_i and f_j are considered as polynomials in y_2.

Since resultants seem to have a better run-time behaviour, this could lead to a speed-up of the algorithm.

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