On the stability of Gröbner bases under specializations

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Let R be a Noetherian commutative ring with identity, K a field and π a ring homomorphism from R to K. We investigate for which ideals in $R[x_1, \ldots, x_n]$ and admissible orders the formation of leading monomial ideals commutes with the homomorphism π.

1. Introduction

Let R, R' be Noetherian commutative rings with identity and $\pi : R \to R'$ a ring homomorphism. When does a Gröbner basis of an ideal $I \subseteq R[x_1, \ldots, x_n]$ map to a Gröbner basis of the ideal $I\, R'[x_1, \ldots, x_n]$ generated by the image of I under the natural extension $\pi : R[x_1, \ldots, x_n] \to R'[x_1, \ldots, x_n]$? Obviously it suffices to have

$$lm(I)\, R'[x_1, \ldots, x_n] = lm(I\, R'[x_1, \ldots, x_n]), \tag{1.1}$$

where $lm(I)$ denotes the ideal generated by the leading monomials of the elements of I. This condition has already been studied in Bayer et al. (1991) and it has been shown that (1.1) holds for any ideal and any term order if and only if π is flat.

In this paper we study condition (1.1) under the additional assumption that R' is not a general Noetherian commutative ring with identity but a field. First we prove the following necessary and sufficient condition for (1.1). Let $\{g_1, \ldots, g_s\}$ be a Gröbner basis of an ideal $I \subseteq R[x_1, \ldots, x_n]$ with respect to an order $\prec$ and assume that the g_i's are ordered in such a way that the leading coefficients of precisely the first r polynomials are not in the kernel $ker(\pi)$. Then (1.1) holds for I and $\prec$ if and only if the polynomials $\pi(g_{r+1}), \ldots, \pi(g_s)$ can be reduced to 0 modulo $\{\pi(g_1), \ldots, \pi(g_r)\}$. Sufficient but not necessary conditions that (1.1) holds for an ideal and an order can be found in Bayer et al. (1991), Pauer (1992), Gräbe (1993) and Assi (1994).

If R' is a field $ker(\pi)$ is a prime ideal. Let J be a subideal of $ker(\pi)$. We show that the following two conditions are equivalent:

(a) $ker(\pi)$ is an isolated prime ideal of J.
(b) For any ideal I in the univariate polynomial ring $R[x]$ with $I \cap R = J$, (1.1) holds.

Furthermore we use the concept of independence complexes of ideals to give two other conditions equivalent to (a) and (b). Note that the implication (a)$\Rightarrow$(b) is a generalization of the main result in Gianni (1987) and Kalkbrener (1987).

For ideals in multivariate polynomial rings over R we prove the equivalence of the following two conditions:

(c) $ker(\pi)$ is an isolated prime ideal of J which equals the corresponding primary component.
(d) For any number of variables n, any ideal I in $R[x_1, \ldots, x_n]$ with $I \cap R = J$ and any term order, (1.1) holds.

As a consequence of this result and the already mentioned theorem in Bayer et al. (1991) we obtain that π is flat if and only if no proper subideal of $ker(\pi)$ is primary.

2. Definitions

Throughout this paper let R be a Noetherian commutative ring with identity and K a field. The ideal generated by a subset F of R is denoted by $\langle F \rangle$ and the set of power products in the variables $x_1, \ldots, x_n$ by $PP(x_1, \ldots, x_n)$. Let $\prec$ be an arbitrary admissible order on $PP(x_1, \ldots, x_n)$. For any non-zero polynomial $f \in R[x_1, \ldots, x_n]$ write $f = cX + f'$, where $c \in R \setminus \{0\}$ and $X \in PP(x_1, \ldots, x_n)$ with $X \succ X'$ for every power product X' in f'. With this notation we set

$$\begin{array}{ll} lc(f) := c, & \text{the leading coefficient of } f, \\ lpp(f) := X, & \text{the leading power product of } f, \\ lm(f) := cX, & \text{the leading monomial of } f. \end{array}$$

The total degree of f in $x_1, \ldots, x_n$ is denoted by $deg(f)$. Furthermore, we define $lc(0) := lpp(0) := lm(0) := 0$ and $deg(0) := -1$. For an ideal I in $R[x_1, \ldots, x_n]$ we denote the ideal $\langle \{lm(f) \mid f \in I\} \rangle$ by $lm(I)$. A finite subset G of an ideal $I \subseteq R[x_1, \ldots, x_n]$ is a Gröbner basis of I w.r.t. $\prec$ if

$$\langle \{lm(g) \mid g \in G\} \rangle = lm(I).$$

We will often use the characterization of Gröbner bases in Theorem 2.1 (see Möller (1988)). Let $F = \{f_1, \ldots, f_r\}$ be a subset of $R[x_1, \ldots, x_n]$ and $M := (lm(f_1), \ldots, lm(f_r))$. A syzygy w.r.t. M is an r-tuple of polynomials $S = (h_1, \ldots, h_r)$ in $R[x_1, \ldots, x_n]^r$ such that

$$\sum_{i=1}^{r} h_i \cdot lm(f_i) = 0.$$

The set $S(M)$ of all syzygies w.r.t. M forms an $R[x_1, \ldots, x_n]$-module. An element $S \in S(M)$ is homogeneous of degree X, where $X \in PP(x_1, \ldots, x_n)$, provided that

$$S = (c_1 Y_1, \ldots, c_r Y_r),$$

where $c_i \in R$, $Y_i \in P(x_1, \ldots, x_n)$ and $Y_i \cdot lpp(f_i) = X$ whenever $c_i \neq 0$. Obviously, $S(M)$ has a finite homogeneous basis.

Theorem 2.1. *Let $F = \{f_1, \ldots, f_r\}$ be a subset of $R[x_1, \ldots, x_n]$ and $M := (lm(f_1), \ldots, lm(f_r))$. The following two conditions are equivalent:*

(a) F is a Gröbner basis of $\langle F \rangle$.

(b) Let $S_1, \ldots, S_m$ be a basis of $S(M)$, $S_i = (h_{i1}, \ldots, h_{ir})$ homogeneous for every $i \in \{1, \ldots, m\}$. Then any polynomial $p_i = \sum_{j=1}^{r} h_{ij} f_j$ can be written in the form $p_i = \sum_{j=1}^{r} g_{ij} f_j$, where the g_{ij} are in $R[x_1, \ldots, x_n]$ and $lpp(p_i) = \max_{j=1}^{r} lpp(g_{ij}) lpp(f_j)$.

Let R' be a Noetherian commutative ring with identity. Every ring homomorphism $\pi : R \to R'$ extends naturally to a homomorphism $\pi : R[x_1, \ldots, x_n] \to R'[x_1, \ldots, x_n]$. The image under π of an ideal $I \subseteq R[x_1, \ldots, x_n]$ generates the extension ideal $I\,R'[x_1, \ldots, x_n]$. We want to study under which conditions on π and $\prec$ a Gröbner basis of I maps to a Gröbner basis of $I\,R'[x_1, \ldots, x_n]$. Note that it suffices to have

$$lm(I)\,R'[x_1, \ldots, x_n] = lm(I\,R'[x_1, \ldots, x_n]). \tag{2.1}$$

We call I stable under π and $\prec$ if it satisfies (2.1) and we will focus on this condition.

The stability of ideals has been already studied by Bayer, Galligo and Stillman. They proved the following interesting relation between flat morphisms and the stability of ideals (Theorem 3.6 in Bayer et al. (1991)). Recall that an R-module N is called flat if the functor $T_N : M \to M \otimes_R N$ on the category of R-modules is exact and the ring homomorphism $\pi : R \to R'$ is called flat if π makes R' a flat R-module.

Theorem 2.2. *Let $\pi : R \to R'$ be a ring homomorphism. Then the following two conditions are equivalent:*

(a) For any natural number n, any ideal I in $R[x_1, \ldots, x_n]$ and any admissible order $\prec$ on $PP(x_1, \ldots, x_n)$, I is stable under π and $\prec$.

(b) π is flat.

In this paper we will concentrate on a special case: we assume that π is a ring homomorphism from R to the field K. Hence the image of R is a subring of K and therefore an integral domain. Thus the kernel $ker(\pi)$ is a prime ideal and the quotient field $\bar{K}$ of $R/ker(\pi)$ is a subfield of K. Furthermore, it is easy to see that

$$\text{the ideal } lm(I\,K[x_1, \ldots, x_n]) \text{ is generated by the set } \{lm(\pi(f)) \mid f \in I\}. \tag{2.2}$$

A subset $\{x_{i_1}, \ldots, x_{i_m}\} \subseteq \{x_1, \ldots, x_n\}$ is called independent modulo an ideal $J \subseteq K[x_1, \ldots, x_n]$ if $J \cap K[x_{i_1}, \ldots, x_{i_m}] = \{0\}$. The independence complex of J is the set

$$\Delta(J) := \{\{x_{i_1}, \ldots, x_{i_m}\} \subseteq \{x_1, \ldots, x_n\} \mid \{x_{i_1}, \ldots, x_{i_m}\} \text{ is independent modulo } J\}.$$

Additionally to stability we will consider the following weaker property. We call an ideal $I \subseteq R[x_1, \ldots, x_n]$ semi-stable under π and $\prec$ if

$$\Delta(lm(I)\,K[x_1, \ldots, x_n]) = \Delta(lm(I\,K[x_1, \ldots, x_n])). \tag{2.3}$$

3. Stability criteria

First of all we show that the stability of an ideal I can be easily checked if a Gröbner basis of I is known.

Theorem 3.1. *Let π be a ring homomorphism from R to K, I an ideal in $R[x_1, \ldots, x_n]$ and $G = \{g_1, \ldots, g_s\}$ a Gröbner basis of I with respect to an admissible order $\prec$. We assume that the g_i's are ordered in such a way that there exists an $r \in \{0, \ldots, s\}$ with $\pi(lc(g_i)) \neq 0$ for $i \in \{1, \ldots, r\}$ and $\pi(lc(g_i)) = 0$ for $i \in \{r+1, \ldots, s\}$. Then the following three conditions are equivalent:*

(a) I is stable under π and $\prec$.

(b) $\{\pi(g_1), \ldots, \pi(g_r)\}$ *is a Gröbner basis of* $I\,K[x_1, \ldots, x_n]$ *w.r.t.* $\prec$.
(c) For every $i \in \{r+1, \ldots, s\}$ *the polynomial* $\pi(g_i)$ *is reducible to 0 modulo* $\{\pi(g_1), \ldots, \pi(g_r)\}$.

Proof: Obviously $\{\pi(g_1), \ldots, \pi(g_r)\}$ is a Gröbner basis of $I\,K[x_1, \ldots, x_n]$ if and only if

$$\langle \{\pi(lm(g)) \mid g \in G\} \rangle = lm(I\,K[x_1, \ldots, x_n]).$$

Since

$$\langle \{\pi(lm(g)) \mid g \in G\} \rangle = lm(I)\,K[x_1, \ldots, x_n]$$

(a) and (b) are equivalent.

If $\{\pi(g_1), \ldots, \pi(g_r)\}$ is a Gröbner basis of $I\,K[x_1, \ldots, x_n]$ then (c) holds. It remains to show that (c) implies (a). Let $f \in I$ with $\pi(f) \neq 0$. By (2.2), it suffices to show that

$$\text{there exists a } g \in I \text{ such that } lpp(g) \text{ divides } lpp(\pi(f)) \text{ and } \pi(lc(g)) \neq 0. \tag{3.1}$$

We do the proof by induction on $\prec$.

Induction basis: If $lpp(f) = 1$ then $\pi(lc(f)) \neq 0$ and $lpp(f) = lpp(\pi(f))$. Hence, (3.1) holds.

Induction step: Since (3.1) holds if $\pi(lc(f)) \neq 0$ we assume that $\pi(lc(f)) = 0$. If there exists an $i \in \{1, \ldots, r\}$ such that $lpp(g_i)$ divides $lpp(f)$ we define

$$f' := lc(g_i) \cdot f - lc(f) \cdot (lpp(f)/lpp(g_i)) \cdot g_i.$$

Obviously, $lpp(\pi(f')) = lpp(\pi(f))$ and $lpp(f') \prec lpp(f)$. Thus, (3.1) follows from the induction hypothesis. Otherwise, there exist $j_1, \ldots, j_k \in \{r+1, \ldots, s\}$ and $c_{j_1}, \ldots, c_{j_k} \in R$ such that $lpp(g_{j_l})$ divides $lpp(f)$ for $l \in \{1, \ldots, k\}$ and

$$lm(f) = \sum_{l=1}^{k} c_{j_l} \cdot (lpp(f)/lpp(g_{j_l})) \cdot lm(g_{j_l}).$$

Let $i \in \{r+1, \ldots, s\}$. Since $\pi(g_i)$ is reducible to 0 modulo $\{\pi(g_1), \ldots, \pi(g_r)\}$ there exist an $h_i \in I$ and a $b_i \in R \setminus ker(\pi)$ with $\pi(b_i) \cdot \pi(g_i) = \pi(h_i)$ and $lpp(g_i) \succ lpp(\pi(g_i)) = lpp(h_i)$. Define

$$f' := b \cdot f - \sum_{l=1}^{k} (b/b_{j_l}) \cdot c_{j_l} \cdot (lpp(f)/lpp(g_{j_l})) \cdot (b_{j_l} \cdot g_{j_l} - h_{j_l}),$$

where $b := \prod_{l=1}^{k} b_{j_l}$. Obviously, $lpp(\pi(f')) = lpp(\pi(f))$ and $lpp(f') \prec lpp(f)$. Again, (3.1) follows from the induction hypothesis. $\square$

Sufficient but not necessary criteria for the stability of I under π and $\prec$ can be found in Bayer et al. (1991), Pauer (1992), Gräbe (1993) and Assi (1994).

Let J be an ideal in R with $J \subseteq ker(\pi)$. We will now show that every ideal I in the univariate polynomial ring $R[x_1]$ with $I \cap R = J$ is stable resp. semi-stable under π if and only if

$$ker(\pi) \text{ is an isolated prime ideal of } J. \tag{3.2}$$

Another condition equivalent to (3.2) is semi-stability of every ideal I in a multivariate polynomial ring over R with $I \cap R = J$.

Theorem 3.2. *Let π be a ring homomorphism from R to K and J an ideal in R with $J \subseteq ker(\pi)$. Then the following four conditions are equivalent:*

(a) $ker(\pi)$ is an isolated prime ideal of J.

(b) For any ideal I in $R[x_1]$ with $I \cap R = J$, I is stable under π and the uniquely determined admissible order $\prec$ on $PP(x_1)$.

(c) For any natural number n, any ideal I in $R[x_1, \ldots, x_n]$ with $I \cap R = J$ and any admissible order $\prec$ on $PP(x_1, \ldots, x_n)$, I is semi-stable under π and $\prec$.

(d) For any ideal I in $R[x_1]$ with $I \cap R = J$, I is semi-stable under π and the uniquely determined admissible order $\prec$ on $PP(x_1)$.

Proof: Denote the kernel of π by P.

$(a) \Rightarrow (c)$: Let I be an ideal in $R[x_1, \ldots, x_n]$ with $I \cap R = J$ and $\prec$ an admissible order on $PP(x_1, \ldots, x_n)$. Assume that P is an isolated prime ideal of J and $f \in I$ with $\pi(f) \neq 0$. We first show that

$$\text{there exists a natural number } l \text{ with } lm(\pi(f))^l \in lm(I)\, K[x_1, \ldots, x_n]. \tag{3.3}$$

Write f in the form $f = a_1X_1 + \ldots + a_tX_t$, where $a_1, \ldots, a_t \in R \setminus \{0\}$ and $X_1, \ldots, X_t \in PP(x_1, \ldots, x_n)$ with $X_1 \succ \ldots \succ X_t$. Choose $k \in \{1, \ldots, t\}$ with $a_1, \ldots, a_{k-1} \in P$ and $a_k \notin P$ and define $p := a_1X_1 + \ldots + a_{k-1}X_{k-1}$ and $h := a_kX_k + \ldots + a_tX_t$. Let $I = Q_1 \cap \ldots \cap Q_m$ be an irredundant primary decomposition of I and denote the radical of Q_i by P_i. We can assume that the Q_i's are ordered in such a way that there exists an $m' \in \{1, \ldots, m\}$ with $P = P_j \cap R$ for $j \in \{1, \ldots, m'\}$ and $P \neq P_j \cap R$ for $j \in \{m'+1, \ldots, m\}$. Obviously, $p, h \in P_j$ for $j \in \{1, \ldots, m'\}$. Hence, we can choose a natural number l such that for every $j \in \{1, \ldots, m'\}$ we have $h^l \in Q_j$. Since P is an isolated prime ideal of $I \cap R$ we can choose for every $j \in \{m'+1, \ldots, m\}$ a $q_j \in (Q_j \cap R) \setminus P$. For $q := \prod_{j=m'+1}^{m} q_j$ we have $qh^l \in I$ and $\pi(lm(qh^l)) = \pi(q) \cdot lm(\pi(f))^l$. Hence, (3.3) is proved.

For proving semi-stability it suffices to show that

$$\Delta(lm(I)\, K[x_1, \ldots, x_n]) \subseteq \Delta(\langle \{lm(\pi(f)) \mid f \in I\} \rangle).$$

Let $\{x_{i_1}, \ldots, x_{i_k}\} \notin \Delta(\langle \{lm(\pi(f)) \mid f \in I\} \rangle)$. Then there exists an $f \in I$ such that $lm(\pi(f)) \in K[x_{i_1}, \ldots, x_{i_k}] \setminus \{0\}$. By (3.3), there exists a natural number l with

$$lm(\pi(f))^l \in (lm(I)\, K[x_1, \ldots, x_n]) \cap K[x_{i_1}, \ldots, x_{i_k}]$$

and therefore $\{x_{i_1}, \ldots, x_{i_k}\} \notin \Delta(lm(I)\, K[x_1, \ldots, x_n])$. Thus, I is semi-stable under π and $\prec$.

$(c) \Rightarrow (b)$: Let I be an ideal in $R[x_1]$ with $I \cap R = J$ and $\prec$ the uniquely determined admissible order on $PP(x_1)$. If $lm(I\, K[x_1]) = \{0\}$ then I is obviously stable under π and $\prec$. Hence, we can assume that $lm(I\, K[x_1])$ is generated by x_1^k for some non-negative integer k. It follows from (c) that $lm(I)\, K[x_1]$ is generated by x_1^l for some non-negative integer l with $k \leq l$. Assume that I is not stable and therefore $k < l$. By (2.2), there exist f_1 and f_2 in I with $deg(\pi(f_1)) = k$ and $deg(f_2) = deg(\pi(f_2)) = l$. Let f_3 be the pseudo-remainder of $x_1^{l-k-1}f_1$ and f_2. Obviously, $l-1 = deg(\pi(x_1^{l-k-1}f_1)) = deg(\pi(f_3))$ and $deg(f_3) < deg(f_2)$. Hence, we obtain $deg(f_3) = deg(\pi(f_3)) = l-1$, a contradiction to the definition of l.

Since (b) implies (d) it remains to show $(d) \Rightarrow (a)$:

Assume that P is not an isolated prime ideal of J. Let $J = Q_1 \cap \ldots \cap Q_m$ be an irredundant primary decomposition of J and denote the radical of Q_i by P_i. We can assume that the Q_i's are ordered in such a way that there exists an $m' \in \{0, \ldots, m-1\}$ with $P \subseteq P_j$ for $j \in \{1, \ldots, m'\}$ and $P \not\subseteq P_j$ for $j \in \{m'+1, \ldots, m\}$. Thus the prime ideal P is not contained in $\bigcup_{j=m'+1}^{m} P_j$ (see Matsumura (1970), p.3). Hence, we can choose an element c of P such that

$$c \in \bigcap_{j=1}^{m'} Q_j \quad \text{and} \quad c \notin \bigcup_{j=m'+1}^{m} P_j.$$

Furthermore, let $\{a_1, \ldots, a_r\}$ be a generating set of J, $\{b_1, \ldots, b_k\}$ a generating set of $Q_{m'+1} \cap \ldots \cap Q_m$ and

$$G := \{a_1, \ldots, a_r, b_1 x_1, \ldots, b_k x_1, c x_1^2 - x_1\}.$$

Obviously, $\langle G \rangle \cap R = J$. We will show that G is a Gröbner basis of $I := \langle G \rangle$. Let $S = (s_1, \ldots, s_r, s_1, \ldots, s_k, s)$ be a homogeneous syzygy w.r.t. the tuple $(a_1, \ldots, a_r, b_1 x_1, \ldots, b_k x_1, c x_1^2)$. Since

$$(Q_{m'+1} \cap \ldots \cap Q_m) : c = Q_{m'+1} \cap \ldots \cap Q_m,$$

the coefficient of s is an element of $Q_{m'+1} \cap \ldots \cap Q_m$. Hence, $s x_1$ is an element of the monomial ideal $\langle \{a_1, \ldots, a_r, b_1 x_1, \ldots, b_k x_1\} \rangle$ and therefore, by Theorem 2.1, G is a Gröbner basis.

We will use this fact in order to show that I is not semi-stable. We have assumed that $J \subseteq P$ and P is not an isolated prime ideal of J. Hence, by definition of m', there exists a $j \in \{m'+1, \ldots, m\}$ with $Q_j \subseteq P_j \subseteq P$. Thus, $\{a_1, \ldots, a_r, b_1, \ldots, b_k, c\} \subseteq P$ and therefore

$$\Delta(lm(I)\, K[x_1]) = \{\{x_1\}, \emptyset\} \neq \{\emptyset\} = \Delta(lm(I\, K[x_1])). \quad \square$$

Note that the implication $(a) \Rightarrow (b)$ in Theorem 3.2 is a generalization of the main result in Gianni (1987) and Kalkbrener (1987).

In Theorem 3.2 we have proved that every ideal I in $R[x_1]$ with $I \cap R = J$ is stable if and only if $ker(\pi)$ is an isolated prime ideal of J. In the following theorem we will give a similar characterization of the stability of multivariate ideals. Note that the implication $(a) \Rightarrow (b)$ in Theorem 3.3 is similar to Proposition 3.10 in Bayer et al. (1991) and a generalization of Theorem 2 in Becker (1994).

THEOREM 3.3. *Let π be a ring homomorphism from R to K and J an ideal in R with $J \subseteq ker(\pi)$. Then the following three conditions are equivalent:*

(a) $ker(\pi)$ is an isolated prime ideal of J which equals the corresponding primary component.

(b) For any natural number n, any ideal I in $R[x_1, \ldots, x_n]$ with $I \cap R = J$ and any admissible order $\prec$ on $PP(x_1, \ldots, x_n)$, I is stable under π and $\prec$.

(c) For any ideal I in $R[x_1, x_2]$ with $I \cap R = J$ and any admissible order $\prec$ on $PP(x_1, x_2)$, I is stable under π and $\prec$.

Proof: Denote the kernel of π by P.

$(a) \Rightarrow (b)$: If P equals the corresponding primary component then it follows from the proof of the previous theorem that we can choose l as 1 in (3.3).

Since (b) implies (c) it remains to show $(c) \Rightarrow (a)$:
If P is not an isolated prime ideal of J it follows from Theorem 3.2 that there exists an ideal I in $R[x_1, x_2]$ which satisfies $I \cap R = J$ and is not semi-stable. Hence, we assume that P is an isolated prime ideal of J which is unequal to the corresponding primary component Q. Let $c \in P$ and $l > 1$ the smallest natural number with $c^l \in Q$. For every non-negative integer j let $B_j = \{b_{j1}, \ldots, b_{ji_j}\}$ be a finite basis of the ideal quotient $J : c^j$. Since $J \subseteq J : c \subseteq J : c^2 \ldots$ is an ascending chain of ideals there exists a natural number r with $J : c^r = J : c^k$ for every $k \geq r$. Define

$$G := \bigcup_{j=0}^{r} \{bx_1^j \mid b \in B_j\} \cup \{cx_2 - x_1\}$$

and $I := \langle G \rangle$. Obviously, $I \cap R = J$. We will now show that G is a Gröbner basis with respect to every admissible order with $x_1 \prec x_2$. Using Theorem 2.1 it suffices to show that for every homogeneous syzygy $S = (s_{11}, \ldots, s_{ri_r}, s)$ w.r.t. the tuple $(b_{11}, \ldots, b_{ri_r}x_1^r, cx_2)$ the monomial sx_1 is an element of the monomial ideal generated by $\bigcup_{j=0}^{r}\{bx_1^j \mid b \in B_j\}$. Let $x_1^{k_1}x_2^{k_2}$ be the degree of S. Obviously, the coefficient of s is an element of the ideal generated by B_{k_1+1} in R. Hence, sx_1 is an element of $\langle\{bx_1^{k_1+1} \mid b \in B_{k_1+1}\}\rangle$ and therefore an element of the ideal generated by $\bigcup_{j=0}^{r}\{bx_1^j \mid b \in B_j\}$.

Since P is an isolated prime ideal of J we have $B_j \subseteq P$ for $j \in \{0, \ldots, l-1\}$ and $B_l \not\subseteq P$. Hence, $lm(I)\,K[x_1, \ldots, x_n] = \{x_1^l\}$ and $lm(I\,K[x_1, \ldots, x_n]) = \{x_1\}$. □

Let I be an ideal in $R[x_1, \ldots, x_n]$ such that $ker(\pi)$ is an isolated prime ideal of $I \cap R$ but unequal to the corresponding primary component. It has been proved in the above theorem that in this case I is not necessarily stable. The next example shows that even the Gröbner basis property may not be preserved for Gröbner bases of I.

EXAMPLE 3.1. Let $\mathbf{Q}$ denote the rational numbers and define $R := \mathbf{Q}[y]$, $K := \mathbf{Q}$. Let π be the natural map from $\mathbf{Q}[y]$ to $\mathbf{Q}[y]/\langle y \rangle$ and I the ideal in $R[x_1, x_2, x_3, x_4]$ generated by

$$\{y^2,\ yx_1,\ x_1^2,\ yx_2 + x_1,\ x_1x_4 + x_3\}.$$

The set

$$G = \{y^2,\ yx_1,\ x_1^2,\ yx_2 + x_1,\ yx_3,\ x_1x_3,\ x_3^2,\ x_1x_4 + x_3\}$$

is a Gröbner basis of I with respect to the lexicographical order $\prec$ with $x_4 \succ x_3 \succ x_2 \succ x_1$. Thus, $I \cap R = \langle\{y^2\}\rangle$ and $ker(\pi) = \langle\{y\}\rangle$ is an isolated prime ideal of $I \cap R$. Obviously, I is semi-stable but not stable under π and $\prec$ and the image of G under π is not a Gröbner basis.

As a consequence of Theorem 2.2 and Theorem 3.3 we obtain the following characterization of flatness.

COROLLARY 3.1. *Let π be a ring homomorphism from R to K.*
(a) The ring homomorphism π is flat iff no proper subideal of the kernel of π is primary.

(b) If $\langle 0\rangle \subseteq R$ *is primary but not prime then* π *is not flat.*

(c) If $\langle 0\rangle \subseteq R$ *is prime then* π *is flat iff the kernel of* π *is* $\langle 0\rangle$*.*

Proof: Denote the kernel of π by P.

(a) Assume that there exists a proper subideal Q of P which is primary. By Theorem 3.3, there exists an ideal $I \subseteq R[x_1,\ldots,x_n]$ and an admissible order $\prec$ such that I is not stable under π and $\prec$. Hence, by Theorem 2.2, π is not flat.

Assume that no proper subideal Q of P is primary and let I be an ideal in $R[x_1,\ldots,x_n]$ and $\prec$ an admissible order. If $I \cap R \not\subseteq P$ then

$$lm(I\,K[x_1,\ldots,x_n]) = \langle 1\rangle = lm(I)\,K[x_1,\ldots,x_n]. \tag{3.4}$$

Otherwise, P is an isolated prime ideal of $I \cap R$ which equals the corresponding primary component. By Theorem 3.3, $lm(I\,K[x_1,\ldots,x_n]) = lm(I)\,K[x_1,\ldots,x_n]$. Together with (3.4) and Theorem 2.2, π is flat.

(b) and (c) follow from (a) immediately. □

Example 3.2. Let $R := \mathbf{Q}[x]/\langle x^2(x-1)\rangle$ and consider the following homomorphisms from R to $\mathbf{Q}$: π_1 is the natural map from R to $\mathbf{Q}[x]/\langle x\rangle$ and π_2 is the natural map from R to $\mathbf{Q}[x]/\langle x-1\rangle$. Then π_2 is flat and π_1 is not.

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