Efficient calculation of expected shortfall contributions in large credit portfolios

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Abstract

In the framework of a standard structural credit portfolio model, we investigate the Monte Carlo based estimation of capital allocation according to expected shortfall. We develop and analyze several variance reduction techniques based on importance sampling, analytic approximations of portfolio loss distributions and a semi-analytical allocation technique. The focus of the paper is on the application of these techniques to large credit portfolios used in economic capital calculations. Our results show that the inherent numerical problems of expected shortfall allocation can be overcome and, as a consequence, economic capital allocation according to expected shortfall is a viable option for financial institutions.

Key words: Monte Carlo simulation, variance reduction, importance sampling, portfolio credit risk, expected shortfall allocation

1 Introduction

In a typical bank, risk capital for credit risk far outweighs capital requirements for any other risk class. Key drivers of credit risk are concentrations in a bank's credit portfolio. These risk concentrations may be caused by material concentrations of exposure to individual names as well as large exposures to single sectors (geographic regions or industries) or to several highly correlated sectors. The most common approach to introduce sector concentration into a credit portfolio model is through systematic factors affecting multiple borrowers. Conditional on the systematic factors, the residual default risks of individual borrowers are considered independent and modeled by specific (or idiosyncratic) risk factors.[3] In this model, the credit worthiness of each borrower is defined by a so-called Ability-to-Pay variable that is

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[3]In this paper, we assume that the systematic and specific factors follow a multi-variate normal distribution as proposed by Gupton et al. (1997) in CreditMetrics. We will refer to this model class as Gaussian multi-factor models. See Crouhy et al. (2000) and Bluhm et al. (2002) for a survey on credit portfolio modeling.

completely specified by the systematic risk factors and the specific risk factor of the borrower. In particular, default and rating of a borrower at the end of the planning period are determined by the value of its Ability-to-Pay variable.[4]

This credit portfolio model captures credit losses due to default and due to rating migration and provides an appropriate framework for assessing credit risk at different levels of the bank.

Top level: quantification of the risk in the bank's credit portfolio, which is usually expressed as the bank's economic capital for credit risk.

Lower levels: economic capital allocation to subportfolios and individual transactions.

The standard approach in the finance industry is to define the economic capital in terms of a quantile of the portfolio loss distribution. The capital charge of an individual transaction is usually based on a covariance technique and called volatility contribution. However, there is theoretical and practical evidence that the combination of quantiles and covariances is not a satisfactory approach to risk measurement and capital allocation in credit portfolios (Kalkbrener et al., 2004).

An alternative definition of economic capital is based on the expected shortfall, which can intuitively be interpreted as the average of all losses above a given quantile of the loss distribution. It is well known that expected shortfall satisfies the axioms of coherent risk measures (Acerbi and Tasche, 2002) proposed in Artzner et al. (1999). Moreover, there is a natural way to allocate the expected shortfall of the portfolio: the expected shortfall contribution of a transaction is its average contribution to the portfolio losses above the specified quantile. It has been demonstrated in Kalkbrener et al. (2004) that expected shortfall allocation detects concentration risk more accurately than covariance techniques.

Despite its theoretical and practical advantages, there is a major obstacle to the application of expected shortfall allocation in Gaussian multi-factor models. Since the loss distributions of the portfolio and single transactions are not tractable in analytical form, Monte Carlo techniques are the standard approach to the actual calculation of expected shortfall contributions. It is easy to see that due to statistical fluctuations the simulation-based estimation of this conditional expectation is a demanding computational problem,[5] in particular for large portfolios.

The objective of this paper is the development of variance reduction techniques for expected shortfall that make it practically feasible to allocate economic capital to individual transactions. The economic capital of a bank is derived from the loss distribution of its entire credit portfolio. Even after the application of segmentation

[4] The multi-state model distinguishes between different ratings of a borrower whereas the two-state model only identifies default and non-default events.

[5] We refer to Kurth and Tasche (2003) for a computational approach to expected shortfall in the analytic framework of CreditRisk+.

techniques, this portfolio may consist of more than 100000 different borrowers. As a consequence, 100 systematic and 100000 specific factors are a realistic set-up for the economic capital calculation in a bank. The focus of this paper is therefore on the development of efficient numerical techniques for expected shortfall allocation that are tailored to portfolios of that size.

A Gaussian multi-factor model with rating migration serves as quantitative framework for the development of the algorithms. An important feature of this model is the large number of independent specific factors. This property can be utilized by splitting the calculation of expected shortfall contributions into two steps (compare to Glasserman and Li (2005) or McNeil et al. (2005) in the more general context of Mixture Models):

1. Simulation of systematic factors.

   Typically, these factors are the main drivers for large portfolio losses. Efficient variance reduction techniques are therefore particularly important for the simulation of the systematic factors.

2. Calculation of expected shortfall contributions in each systematic scenario.

   Conditional on a systematic scenario, loss variables of individual borrowers are independent. There exist several options how to exploit conditional independence for stabilizing expected shortfall contributions.

We use an importance sampling technique to improve the Monte Carlo simulation of the systematic factors. In the credit risk literature, importance sampling has been recently suggested in a number of papers, see Avranitis and Gregory (2001), Glasserman and Li (2005), Kalkbrener et al. (2004), Merino and Nyfelder (2004), Morokoff (2004), Glasserman (2005), Egloff et al. (2005) and Glasserman et al. (2007). In contrast to straightforward Monte Carlo simulation, importance sampling puts more weight on the sample range of interest, thereby making the simulation more efficient. However, it is generally far from obvious how such a change of measure should be obtained in a practical manner. In a Gaussian multi-factor model, a natural importance sampling measure is a negative shift of the systematic factors: a negative shift enforces a higher number of defaults and therefore increases the stability of the MC estimate of expected shortfall. For calculating the shift, Glasserman and Li (2005) minimize an upper bound on the second moment of the importance sampling estimator of the tail probability. Furthermore, they show that the corresponding importance sampling scheme is asymptotically optimal. Glasserman et al. (2007) use large deviation analysis to calculate multiple mean shifts. Egloff et al. (2005) suggest an adaptive importance sampling technique that uses the Robbins-Monro stochastic approximation method. Our approach is based on the infinite granularity approximation of the portfolio loss distribution (compare to Vasicek (2002) and Gordy (2003)). More precisely, we approximate the original portfolio P by a homogeneous and infinitely granular portfolio $\bar{P}$. The loss distribution of $\bar{P}$ can be specified by a Gaussian single-factor model. The calculation of the shift of the systematic factors is now done in two steps: in the first step, we calculate the optimal mean in this single-factor setting and then lift the scalar mean to a mean vector for the systematic factors in the original multi-factor model. The efficiency of the proposed importance sampling scheme clearly depends on the quality of the infinite granularity approximation. By definition, the analytic loss distribution of the infinitely granular portfolio provides an excellent fit to portfolio loss distributions of large and well-diversified portfolios. Since these portfolio characteristics are typical for the credit portfolio of a large international bank, we experienced significant improvements in the stability of the economic capital calculations: applied to a realistic test portfolio of 25000 loans it reduces the variance of the Monte Carlo estimate of expected shortfall - and therefore the number of required simulations - by a factor of 400. The average variance reduction experienced for expected shortfall contributions of individual loans is of the order of 150.

The second class of variance reduction techniques presented in this paper utilizes the independence of specific risk factors. We have combined different approaches with importance sampling of systematic factors:

1. importance sampling of specific factors based on exponential twisting of default probabilities (Glasserman and Li, 2005; Merino and Nyfeler, 2004),
2. analytic approximations of conditional loss distributions motivated by the application of the central limit theorem and
3. deterministic calculation of the expected shortfall contribution of the i-th borrower in scenarios where values of all systematic factors and all but the i-th specific factor have been simulated (compare to Merino and Nyfeler (2004)).

The 25000 loan portfolio and a smaller portfolio of 1000 loans have been used for comparing the variance reductions for MC estimates of expected shortfall contributions obtained by these techniques. Our results indicate that the last approach, i.e. importance sampling of systematic factors together with conditional allocation, is particularly well suited for large portfolios: on average, the variance of expected shortfall contributions of individual loans is reduced by a factor of 4000.

The paper has the following structure. For the sake of simplicity we initially develop all variance reduction techniques in a Gaussian multi-factor model that does not distinguish between different rating states but only between default and non-default. This basic two-state model is presented in Section 2.1. Section 2.2 reviews analytic techniques for approximating the portfolio loss distribution in the two-state model. Expected shortfall allocation is formally introduced in Section 3. Section 4 is devoted to the development of the importance sampling technique for systematic risk factors. Variance reduction techniques are applied to specific risk factors in Section 5: importance sampling, conditional expected shortfall allocation and Gaussian approximation techniques. Numerical results are presented in Section 6. Section 7 introduces the rating migration model and generalizes the proposed techniques to this multi-state framework.

2 Loss distributions of Gaussian multi-factor models

2.1 The two-state credit portfolio model

For the sake of simplicity we develop and analyze variance reduction techniques in the framework of a credit portfolio model in default-only mode. We refer to Section 7 for generalizations to models which incorporate rating migration.

The credit portfolio P consists of n loans. With each loan we associate an Ability-to-Pay variable $A_i : \mathbb{R}^{m+1} \to \mathbb{R}$, which is a linear combination of the m systematic variables $x_1, \ldots, x_m$ and a specific variable z_i:

$$A_i(x_1, \ldots, x_m, z_i) := \sum_{j=1}^{m} \phi_{ij} x_j + \sqrt{1 - R_i^2}\, z_i \tag{1}$$

with $0 \leq R_i^2 \leq 1$ and weight vector $(\phi_{i1}, \ldots, \phi_{im})$. The loan loss $L_i : \mathbb{R}^{m+1} \to \mathbb{R}$ and the portfolio loss function $L : \mathbb{R}^{m+n} \to \mathbb{R}$ are defined by

$$L_i := l_i \cdot \mathbf{1}_{\{A_i \leq D_i\}}, \quad L := \sum_{i=1}^{n} L_i, \tag{2}$$

where $0 < l_i$ and $D_i \in \mathbb{R}$ are the (deterministic) loss-at-default and the default threshold respectively. As probability measure $\mathbb{P}$ on $\mathbb{R}^{m+n}$ we use the product measure

$$\mathbb{P} := N_{\mathbf{0},C} \times \prod_{i=1}^{n} N_{0,1},$$

where $N_{0,1}$ is the standardized one-dimensional normal distribution and $N_{\mathbf{0},C}$ the m-dimensional normal distribution with mean $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^m$ and non-singular covariance matrix $C \in \mathbb{R}^m_m$. Note that each x_i, z_i and A_i is a centered and normally distributed random variable under $\mathbb{P}$. We assume that the weight vector $(\phi_{i1}, \ldots, \phi_{im})$ has been normalized in such a way that the variance of A_i is 1. Hence, the default probability p_i of the i-th loan equals

$$p_i := \mathbb{P}(A_i \leq D_i) = N(D_i),$$

where N denotes the standardized one-dimensional normal distribution function. This relation is used to determine the default threshold from empirical default probabilities.

2.2 Analytic approximations

The portfolio loss distribution L defined in (2) can be considered as a discrete distribution on a high dimensional state space with 2^n default/non-default states. It does not have an analytic form. Monte Carlo simulation is the standard technique for the actual calculation of risk capital at portfolio and transaction level. However, Monte Carlo estimates of risk measures derived from the tail of the distribution - like value-at-risk or expected shortfall - tend to be numerically unstable in this credit portfolio model. Analytic approximations have therefore been proposed

1. to calculate portfolio risk and risk contributions in a purely analytical way or
2. for the development of importance sampling techniques in order to improve Monte Carlo stability.

The importance sampling technique proposed in this paper utilizes the infinite granularity approximation of loss distributions of homogeneous portfolios.

2.2.1 Infinite granularity approximation for homogeneous portfolios

Let $\chi = (\chi_1, \ldots, \chi_m) \in \mathbb{R}^m$ be values of the m systematic variables. The specific variables z_i are independent and therefore the L_i are independent on $\{x = \chi\}$. Hence, if n is sufficiently large the portfolio loss function $L = \sum_{i=1}^n L_i$ can be approximated on $\{x = \chi\}$ by applying a limit theorem to L/s, where s is an appropriate scaling factor. The most straightforward approximation is based on applying the law of large numbers to $(1/n) \cdot \sum_{i=1}^n L_i$, i.e. the portfolio loss function L is approximated by its conditional mean on $\{x = \chi\}$.

Consider now a homogeneous portfolio $\bar{P}$, i.e. each loan has the same loss-at-default l, default probability p, R^2 and set of factor weights $(\rho_1, \ldots, \rho_m) \in \mathbb{R}^m$. The application of the above strategy leads to the following result.[6]

Theorem 1 *Let the loss function $\bar{L}_i$ of the i-th loan in the homogeneous portfolio $\bar{P}$ be defined by*

$$\bar{L}_i := l \cdot \mathbf{1}_{\{\bar{A}_i \leq N^{-1}(p)\}},$$

where $\bar{A}_i$ denotes the i-th homogeneous Ability-to-Pay variable

$$\bar{A}_i(x_1, \ldots, x_m, z_i) := \sum_{j=1}^m \rho_j x_j + \sqrt{1 - R^2} z_i.$$

Then

$$\lim_{n \to \infty} (1/n) \cdot \sum_{i=1}^n \bar{L}_i \ = \ l \cdot N\left(\frac{N^{-1}(p) - \sum_{j=1}^m \rho_j x_j}{\sqrt{1 - R^2}}\right)$$

[6] We refer to Vasicek (1991) for a proof. Generalizations are given in Bluhm et al. (2002) and McNeil et al. (2005).

holds almost surely on Ω.

Note that if the linear sum $\sum_{j=1}^m \rho_j x_j$ of the systematic variables in the homogeneous m-factor model is considered as one systematic factor then the m-factor model is transformed into a one-factor model with Ability-to-Pay variables

$$\sqrt{R^2}x + \sqrt{1 - R^2}z_i.$$

In order to utilize this analytic distribution in our inhomogeneous multi-factor setting, the original portfolio P has to be approximated by a homogeneous and infinitely granular portfolio $\bar{P}$. However, there is no unique procedure to establish the homogeneous portfolio, which is closest to a given portfolio.

We propose the following technique for determining the parameters of the homogeneous portfolio $\bar{P}$, i.e. loss-at-default l, default probability p, R^2 and factor weights ρ_j, $j = 1, \ldots, m$:

Loss and default probability. The homogeneous loss l is the average of the individual losses l_i and the homogeneous default probability p is the loss-at-default weighted default probability of all loans in the portfolio:

$$l := \frac{\sum_{i=1}^n l_i}{n}, \qquad p := \frac{\sum_{i=1}^n p_i l_i}{\sum_{i=1}^n l_i}. \tag{3}$$

Weight vector. The homogeneous weight vector is the normalized, weighted sum of the weight vectors of the individual loans. In this paper, the positive weights $g_1, \ldots, g_n \in \mathbb{R}$ are given by $g_i := \mathrm{E}(L_i) = p_i l_i$, i.e. the i-th weight equals the i-th expected loss, and the homogeneous weight vector $\rho = (\rho_1, \ldots, \rho_m)$ is defined by

$$\rho := \psi/s \quad \text{with} \quad \psi = (\psi_1, \ldots, \psi_m) := \sum_{i=1}^n g_i \cdot (\phi_{i1}, \ldots, \phi_{im}). \tag{4}$$

The scaling factor $s \in \mathbb{R}$ is chosen such that

$$R^2 = \sum_{i,j=1}^m \rho_i \cdot \rho_j \cdot \mathrm{Cov}(x_i, x_j) \tag{5}$$

holds, where R^2 is defined in (6).

$\boldsymbol{R^2}$. The specification of the homogeneous R^2 is based on the condition that the weighted sum of Ability-to-Pay covariances is identical in the original and the homogeneous portfolio. More precisely, define

$$R^2 := \frac{\sum_{k,l=1}^m \psi_k \psi_l \mathrm{Cov}(x_k, x_l) - \sum_{i=1}^n g_i^2 R_i^2}{(\sum_{i=1}^n g_i)^2 - \sum_{i=1}^n g_i^2} \tag{6}$$

and the i-th homogeneous Ability-to-Pay variable by

$$\bar{A}_i(x_1, \ldots, x_m, z_i) := \sum_{j=1}^{m} \rho_j x_j + \sqrt{1 - R^2} z_i.$$

The specification of the homogeneous R^2 is motivated by the following result.

Proposition 1 *Equality (7) holds for the weighted sum of Ability-to-Pay covariances of the original and the homogeneous portfolio:*

$$\sum_{i,j=1}^{n} g_i g_j \mathrm{Cov}(A_i, A_j) = \sum_{i,j=1}^{n} g_i g_j \mathrm{Cov}(\bar{A}_i, \bar{A}_j). \tag{7}$$

Proof: We have

$$\begin{aligned} \sum_{i,j=1}^{n} g_i g_j \mathrm{Cov}(A_i, A_j) &= \sum_{i,j=1}^{n} \sum_{k,l=1}^{m} g_i \phi_{ik} g_j \phi_{jl} \mathrm{Cov}(x_k, x_l) + \sum_{i=1}^{n} g_i^2 (1 - R_i^2) \\ &= \sum_{k,l=1}^{m} \psi_k \psi_l \mathrm{Cov}(x_k, x_l) + \sum_{i=1}^{n} g_i^2 (1 - R_i^2) \end{aligned} \tag{8}$$

and, by (5),

$$\begin{aligned} \sum_{i,j=1}^{n} g_i g_j \mathrm{Cov}(\bar{A}_i, \bar{A}_j) &= \sum_{i,j=1}^{n} g_i g_j \sum_{k,l=1}^{m} \rho_k \rho_l \mathrm{Cov}(x_k, x_l) + \sum_{i=1}^{n} g_i^2 (1 - R^2) \\ &= \sum_{i,j=1}^{n} g_i g_j R^2 + \sum_{i=1}^{n} g_i^2 (1 - R^2). \end{aligned} \tag{9}$$

If R^2 is defined by (6) then (8) equals (9) and the proposition is proved. $\square$

Alternative approaches to portfolio homogenization are proposed in Glasserman (2004). He presents two homogeneous single-factor approximations based on

1. matching mean and variance of the loss distributions or

2. approximations of the decay rate of the distribution tail $\mathbb{P}(L > c)$.

Compared to Glasserman's techniques, the heuristic (6) has the advantage that it is extremely fast, even for large portfolios. We refer to Glasserman (2004) for a comparison of the three techniques.

2.2.2 Moment generating functions and saddle-points

Saddle-point approximations are another frequently used analytical technique. Following Glasserman and Li (2005), we have implemented an importance sampling technique for specific factors that uses saddle-points (see Section 5.1). Here we briefly review the definition of saddle-points and their calculation in the portfolio model introduced in Section 2.1.

For a random variable X, the cumulant generating function (KGF) is defined as

$$\psi_X(\theta) = \log(E(e^{\theta X}))$$

for complex θ. The tail probability of X can be recovered from the KGF by a contour integral

$$\mathbb{P}(X > c) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{e^{\psi_X(\theta) - \theta c}}{\theta} \, d\theta, \tag{10}$$

in which the path of integration is up the imaginary axis and runs to the right of the origin to avoid the pole there.

The KGF is a useful construction because when independent random variables are added, their KGFs are added. This feature is important for calculating the KGF

$$\psi_L(\theta, \chi) := \psi_{L|x=\chi}(\theta)$$

of the portfolio loss distribution L conditional on given values $\chi = (\chi_1, \ldots, \chi_m) \in \mathbb{R}^m$ of the systematic factors: on $\{x = \chi\}$, the Ability-to-Pay variables $A_1, \ldots, A_n$ become independent with conditional default probabilities

$$p_i(\chi) := \mathbb{P}(A_i \leq D_i | x = \chi) = N\left(\frac{N^{-1}(p_i) - \sum_{j=1}^{m} \phi_{ij}\chi_j}{\sqrt{1 - R_i^2}} \right) \tag{11}$$

and therefore

$$\psi_L(\theta, \chi) = \sum_{i=1}^{n} \psi_{L_i}(\theta, \chi) = \sum_{i=1}^{n} \log(1 + p_i(\chi)(e^{\theta \, l_i} - 1)).$$

On the real axis, $\psi_L(\theta, \chi) - \theta c$ has a unique minimum $\theta_c(\chi)$, the saddle-point, that can be written as

$$\theta_c(\chi) = \begin{cases} \text{unique } \theta \text{ such that } \frac{\partial}{\partial \theta} \psi_L(\theta, \chi) = c & \text{if} \quad c > E(L | x = \chi), \\ 0 & \text{if} \quad c \leq E(L | x = \chi). \end{cases}$$

Referring to equation (10), Martin et al. (2001a, 2001b) approximate $\mathbb{P}(L > c)$ by a Taylor series expansion of $\psi_L(\theta, \chi) - \theta c$ around the saddle-point $\theta_c(\chi)$. They show that this technique works very well for particular classes of credit portfolio models. However, for Gaussian multi-factor models, this approach would require the calculation of a multidimensional integral whose dimension corresponds to the number of systematic factors. For our purposes, this is a hopeless task (Martin et al., 2001a).

Application of saddle-point techniques in the one-factor Vasicek model can be found in Huang et al. (2006). Glasserman and Li (2005) use saddle-points in their importance sampling for systematic as well as specific factors. More details are given in Section 5.1.

3 Coherent risk measurement and capital allocation

The objective of this section is the formal definition of risk measures and allocation schemes, in particular expected shortfall allocation.

After JP Morgan made its RiskMetrics system public in 1994 value-at-risk became the dominant concept for risk measurement. The value-at-risk $\mathrm{VaR}_\alpha(L)$ of L at level $\alpha \in (0,1)$ is defined as an α-quantile of L. More precisely, in this paper

$$\mathrm{VaR}_\alpha(L) := \inf\{x \in \mathbb{R} \mid \mathbb{P}(L \le x) \ge \alpha\}$$

is the smallest α-quantile. While the VaR methodology encourages diversification for the special case of an elliptically distributed random vector (X, Y), i.e.

$$\mathrm{VaR}(X+Y) \le \mathrm{VaR}(X) + \mathrm{VaR}(Y) \tag{12}$$

(McNeil et al. 2005), in general subadditivity (12) does not hold for value-at-risk. Since for typical credit portfolios the assumption of an elliptical distribution cannot be maintained, diversification, which is commonly considered as a way to reduce risk, may increase value-at-risk.

Another disadvantage of value-at-risk is that the allocation of portfolio VaR to subportfolios and individual transactions is difficult in credit portfolio models with discrete loss distributions (Kalkbrener, 2005). The standard solution is to allocate portfolio VaR proportional to the covariances

$$\mathrm{Cov}(L_1, L), \ldots, \mathrm{Cov}(L_n, L). \tag{13}$$

This allocation technique, called volatility allocation, is the natural choice in classical portfolio theory where portfolio risk is measured by standard deviation (or volatility).

In general, combining volatility allocation with value-at-risk works well as long as all loss distributions are close to normal. However, for credit portfolios it does not: the capital allocated to a subportfolio $\bar{P}$ of P might be greater than the risk capital of $\bar{P}$ considered as a stand-alone portfolio, the capital charge of a loan might even be higher than its exposure (Kalkbrener et al., 2004).

An alternative risk measure is expected shortfall (see, for instance, Rockafellar and Uryasev, 2000; Acerbi and Tasche, 2002): the expected shortfall of L at level α is defined by

$$\mathrm{ES}_\alpha(L) := (1-\alpha)^{-1}\int_\alpha^1 \mathrm{VaR}_u(L)du.$$

An equivalent definition of expected shortfall is

$$\mathrm{ES}_\alpha(L) = (1-\alpha)^{-1}(\mathrm{E}(L\,\mathbf{1}_{\{L>\mathrm{VaR}_\alpha(L)\}}) + \mathrm{VaR}_\alpha(L)\cdot(\mathbb{P}(L\le \mathrm{VaR}_\alpha(L))-\alpha)). \quad (14)$$

It is easy to see that for most loss distributions the expected shortfall ES_α is dominated by the first term

$$\mathrm{E}(L|L>\mathrm{VaR}_\alpha(L)) = (1-\alpha)^{-1}\mathrm{E}(L\cdot\mathbf{1}_{\{L>\mathrm{VaR}_\alpha(L)\}}). \quad (15)$$

Intuitively, expected shortfall can therefore be interpreted as the average of all losses above a given quantile of the loss distribution. The second term in (14) takes care of jumps of the loss distribution at its quantile and ensures coherence as defined in Artzner et al. (1997, 1999). In particular, the subadditivity property (12) holds for expected shortfall.

Another important advantage of expected shortfall is the simple allocation of risk capital to subportfolios or individual transactions: in accordance with (14) the expected shortfall contribution of the i-th loan is defined as

$$\mathrm{ESC}_\alpha(L_i, L) := (1-\alpha)^{-1}(\mathrm{E}(L_i\cdot\mathbf{1}_{\{L>\mathrm{VaR}_\alpha(L)\}}) + \beta_L\cdot\mathrm{E}(L_i\cdot\mathbf{1}_{\{L=\mathrm{VaR}_\alpha(L)\}})) \quad (16)$$

with

$$\beta_L := \frac{\mathbb{P}(L\le\mathrm{VaR}_\alpha(L))-\alpha}{\mathbb{P}(L=\mathrm{VaR}_\alpha(L))}.$$

Again the definition (16) is usually dominated by its first term

$$\mathrm{E}(L_i|L>\mathrm{VaR}_\alpha(L)) = (1-\alpha)^{-1}\mathrm{E}(L_i\cdot\mathbf{1}_{\{L>\mathrm{VaR}_\alpha(L)\}}). \quad (17)$$

Hence, the expected shortfall contribution of a loan can be considered as its average contribution to portfolio losses above quantile $\mathrm{VaR}_\alpha(L)$.

4 Importance sampling applied to systematic factors

Monte Carlo simulation is the standard technique for the actual calculation of expected shortfall at portfolio and transaction level in the Gaussian multi-factor model presented in Section 2.1. The main practical problem in applying expected shortfall to realistic credit portfolios is the computation of numerically stable MC estimates. In the rest of the paper, we present techniques to reduce the variance of Monte Carlo simulation:

1. in this section, importance sampling is applied to the Monte Carlo simulation of systematic factors,

2. in Section 5, variance reduction techniques are developed that utilize the independence of loss variables of individual borrowers conditional on a systematic scenario.

4.1 Straightforward Monte Carlo simulation

The efficient computation of expected shortfall (15) and expected shortfall contributions (17) is a challenging task for realistic portfolios and high confidence levels α. Straightforward Monte Carlo simulation does not work well due to the high variance of $L \cdot \mathbf{1}_{\{L>\mathrm{VaR}_\alpha(L)\}}$ and $L_i \cdot \mathbf{1}_{\{L>\mathrm{VaR}_\alpha(L)\}}$ respectively (see (21) and Sections 4.5 and 6). As an example, assume that we want to compute expected shortfall with respect to the $\alpha = 99.9\%$ quantile and compute $\nu = 100000$ MC samples $s_1 \geq s_2 \geq \ldots \geq s_\nu$ of the portfolio loss L. Then $\mathrm{ES}_\alpha(L)$ becomes

$$(1-\alpha)^{-1}\mathrm{E}(L \cdot \mathbf{1}_{\{L>c\}}) = (1-\alpha)^{-1} \int L \cdot \mathbf{1}_{\{L>c\}} \, d\mathbb{P} = \sum_{i=1}^{100} s_i/100, \tag{18}$$

where $c := \mathrm{VaR}_\alpha(L)$. Since the computation of $\mathrm{ES}_\alpha(L)$ is only based on 100 samples it is subject to large statistical fluctuations and numerically unstable. This is even more true for expected shortfall contributions of individual loans. A significantly higher number of samples has to be computed which makes straightforward MC simulation impracticable for large credit portfolios.

4.2 Monte Carlo simulation based on importance sampling

Importance sampling is a technique for reducing the variance of MC simulations and - as a consequence - the number of samples required for stable results. In our setting, the integral in (18) is replaced by the equivalent integral on the right-hand side of the equation

$$\int L \cdot \mathbf{1}_{\{L>c\}} \, d\mathbb{P} = \int L \cdot \mathbf{1}_{\{L>c\}} \cdot f \, d\bar{\mathbb{P}}, \tag{19}$$

where $\mathbb{P}$ is continuous with respect to the probability measure $\bar{\mathbb{P}}$ and has density f. The objective is to choose $\bar{\mathbb{P}}$ in such a way that the variance of the Monte Carlo estimate for the integral (19) is minimal under $\bar{\mathbb{P}}$. This MC estimate is

$$\mathrm{ES}_\alpha(L)_{\nu,\bar{\mathbb{P}}} := \frac{1}{\nu} \sum_{i=1}^{\nu} L_{\bar{\mathbb{P}}}(i) \cdot \mathbf{1}_{\{L_{\bar{\mathbb{P}}}(i)>c\}} \cdot f(i), \tag{20}$$

where $L_{\bar{\mathbb{P}}}(i)$ is a realization of the portfolio loss L under the probability measure $\bar{\mathbb{P}}$ and $f(i)$ is the corresponding value of the density function.

By the strong law of large numbers and the central limit theorem, $\mathrm{ES}_\alpha(L)_{\nu,\bar{\mathbb{P}}}$ converges to (19) almost surely as $\nu \to \infty$ and the sampling error converges as

$$\sqrt{\nu} \cdot (\mathrm{ES}_\alpha(L)_{\nu,\bar{\mathbb{P}}} - \int L \cdot \mathbf{1}_{\{L>c\}} \, d\mathbb{P}) \xrightarrow{d} N(0, \sigma_{\mathrm{ES}_\alpha(L)}(\bar{\mathbb{P}})), \tag{21}$$

where $\sigma^2_{\mathrm{ES}_\alpha(L)}(\bar{\mathbb{P}})$ is the variance of $L \cdot \mathbf{1}_{\{L>c\}} \cdot f$ under $\bar{\mathbb{P}}$, that is:

$$\sigma^2_{\mathrm{ES}_\alpha(L)}(\bar{\mathbb{P}}) = \int \left(L \cdot \mathbf{1}_{\{L>c\}} \cdot f\right)^2 \, d\bar{\mathbb{P}} - \left(\int L \cdot \mathbf{1}_{\{L>c\}} \, d\mathbb{P}\right)^2. \tag{22}$$

In the following we restrict the set of probability measures $\bar{\mathbb{P}}$, which we consider to determine a minimum of (22): for every $M = (M_1, \ldots, M_m) \in \mathbb{R}^m$ define the probability measure $\mathbb{P}_M$ by[7]

$$\mathbb{P}_M := N_{M,C} \times \prod_{i=1}^{n} N_{0,1}, \tag{23}$$

where $N_{M,C}$ is the m-dimensional normal distribution with mean M and covariance matrix C. In other words, those probability measures are considered which only change the mean of the systematic components $x_1, \ldots, x_m$ in the definition of the Ability-to-Pay variables $A_1, \ldots, A_n$. This choice is motivated by the nature of the problem. The MC estimate (20) can be improved by increasing the number of scenarios that lead to high portfolio losses, i.e. portfolio losses above threshold c. This can be realized by generating a sufficiently large number of defaults in each sample. Since defaults occur when Ability-to-Pay variables fall below default thresholds we can enforce a high number of defaults by adding a negative mean to the systematic components.

Having thus restricted importance sampling to measures of the form (23) we consider $\sigma^2_{\mathrm{ES}_\alpha(L)}$ as a function from $\mathbb{R}^m$ to $\mathbb{R}$ and rephrase

The Variance Reduction Problem: compute a minimum $M = (M_1, \ldots, M_m)$ of the variance

$$\sigma^2_{\mathrm{ES}_\alpha(L)}(M) = \int \left(L \cdot \mathbf{1}_{\{L>c\}} \cdot \frac{n_{\mathbf{0},C}}{n_{M,C}}\right)^2 d\mathbb{P}_M - \left(\int L \cdot \mathbf{1}_{\{L>c\}} \, d\mathbb{P}\right)^2 \tag{24}$$

in $\mathbb{R}^m$, where $n_{\mathbf{0},C}$ and $n_{M,C}$ denote the probability density functions of $N_{\mathbf{0},C}$ and $N_{M,C}$ respectively.

We can formulate the minimization condition as

$$\partial_{M_i} \sigma^2_{\mathrm{ES}_\alpha(L)}(M) = 0, \ \ \forall \, i = 1, \ldots, m. \tag{25}$$

However, for realistic portfolios with thousands of loans this system is analytically and numerically intractable.

[7] Note that the initial measure $\mathbb{P}$ equals $\mathbb{P}_{\mathbf{0}}$.

4.3 Approximation by a homogeneous portfolio

To progress we therefore approximate the original portfolio P by a homogeneous and infinitely granular portfolio $\bar{P}$ as described in Section 2.2. Based on Theorem 1 we define the function $L^\infty : \mathbb{R} \to \mathbb{R}$ by

$$L^\infty(x) := n \cdot l \cdot N\left(\frac{N^{-1}(p) - x}{\sqrt{1 - R^2}}\right) \tag{26}$$

and approximate the portfolio loss function $L(x_1, \ldots, x_m, z_1, \ldots, z_n)$ of the original portfolio P by the loss function

$$L_m^\infty(x_1, \ldots, x_m) := L^\infty\left(\sum_{j=1}^{m} \rho_j x_j\right) \tag{27}$$

of the homogeneous and infinitely granular portfolio. The threshold $c^\infty := \mathrm{VaR}_\alpha(L_m^\infty)$ is defined as the α-quantile of L_m^∞ with respect to the m-dimensional Gaussian measure $N_{\mathbf{0},C}$ and $\sigma^2_{\mathrm{ES}_\alpha(L_m^\infty)}(M)$ denotes the variance of

$$L_m^\infty \cdot \mathbf{1}_{\{L_m^\infty > c^\infty\}} \cdot \frac{n_{\mathbf{0},C}}{n_{M,C}}$$

with respect to $N_{M,C}$. By approximating the finite inhomogeneous portfolio P by an infinite homogeneous portfolio we have transformed the variance reduction problem (24) to

The Variance Reduction Problem for Infinite Homogeneous Portfolios: compute a minimum $M = (M_1, \ldots, M_m)$ of the variance

$$\sigma^2_{\mathrm{ES}_\alpha(L_m^\infty)}(M) = \int \left(L_m^\infty \cdot \mathbf{1}_{\{L_m^\infty > c^\infty\}} \cdot \frac{n_{\mathbf{0},C}}{n_{M,C}}\right)^2 dN_{M,C} - \left(\int L_m^\infty \cdot \mathbf{1}_{\{L_m^\infty > c^\infty\}} \, dN_{\mathbf{0},C}\right)^2 \tag{28}$$

in $\mathbb{R}^m$.

Note that we have achieved a significant reduction of complexity: the dimension of the underlying probability space has been reduced from $m + n$ to m and the loss function L_m^∞ is not a large sum but has a concise analytic form. We emphasize, however, that this approximation technique is only used for determining a mean vector M for importance sampling. The actual calculations of expected shortfall and expected shortfall contributions are based on Monte Carlo simulation of the full portfolio model as specified in Section 2.1.

In the next subsection we will present a simple and efficient algorithm which solves the variance reduction problem for infinite homogeneous portfolios with arbitrary precision.

4.4 Optimal mean for infinite homogeneous portfolios

The computation of the minimum of (28) is done in two steps:

One-factor model: Instead of m systematic factors $x_1, \dots, x_m$ we consider the corresponding one-factor model and compute the minimum $\mu^{(1)} \in \mathbb{R}$ of (28) in the case $m = 1$. We will show that $\mu^{(1)}$ is the minimum of

$$\int_{-\infty}^{N^{-1}(1-\alpha)} \frac{(L_1^\infty \cdot n_{0,1})^2}{n_{M,1}}\, dx.$$

Multi-factor model: The one-dimensional minimum $\mu^{(1)}$ can be lifted to the m-dimensional minimum $\mu^{(m)} = (\mu_1^{(m)}, \dots, \mu_m^{(m)})$ of (28) by

$$\mu_i^{(m)} := \frac{\mu^{(1)} \cdot \sum_{j=1}^m \mathrm{Cov}(x_i, x_j) \cdot \rho_j}{\sqrt{R^2}}. \tag{29}$$

The one-factor model

If the linear sum $\sum_{j=1}^m \rho_j x_j$ of the systematic variables in the homogeneous m-factor model is considered as one systematic factor then the m-factor model is transformed into a one-factor model with Ability-to-Pay variables

$$\sqrt{R^2}x + \sqrt{1 - R^2} z_i$$

and analytic approximation of the portfolio loss function

$$L_1^\infty(x) := L^\infty(\sqrt{R^2}x).$$

Equation (5) implies that the threshold $c^\infty = \mathrm{VaR}_\alpha(L_m^\infty)$ equals $\mathrm{VaR}_\alpha(L_1^\infty)$, the α-quantile of L_1^∞ with respect to the one-dimensional Gaussian distribution $N_{0,1}$. Since $L_1^\infty(x)$ is monotonous,

$$\{x \in \mathbb{R} \mid L_1^\infty(x) > c^\infty\} = \{x \in \mathbb{R} \mid x < N^{-1}(1-\alpha)\}.$$

Hence, the variance $\sigma^2_{\mathrm{ES}_\alpha(L_1^\infty)}(M)$ of

$$L_1^\infty \cdot \mathbf{1}_{\{L_1^\infty > c^\infty\}} \cdot \frac{n_{0,1}}{n_{M,1}}$$

under $N_{M,1}$ can be written as

$$\begin{aligned}
\sigma^2_{\mathrm{ES}_\alpha(L_1^\infty)}(M) &= \int \frac{(L_1^\infty \cdot \mathbf{1}_{\{L_1^\infty > c^\infty\}} \cdot n_{0,1})^2}{n_{M,1}}\, dx - \left(\int L_1^\infty \cdot \mathbf{1}_{\{L_1^\infty > c^\infty\}} \cdot n_{0,1}\, dx \right)^2 \\
&= \int_{-\infty}^{N^{-1}(1-\alpha)} \frac{(L_1^\infty \cdot n_{0,1})^2}{n_{M,1}}\, dx - \left(\int_{-\infty}^{N^{-1}(1-\alpha)} L_1^\infty \cdot n_{0,1}\, dx \right)^2.
\end{aligned} \tag{30}$$

Since the second integral in (30) does not depend on M, it suffices to compute a minimum $\mu^{(1)}$ of

$$\int_{-\infty}^{N^{-1}(1-\alpha)} \frac{(L_1^\infty \cdot n_{0,1})^2}{n_{M,1}}\, dx.$$

This can be easily done by applying numerical techniques.

The multi-factor model

In order to solve the minimization problem (28), $\mu^{(1)}$ has to be transformed into an m-dimensional vector. The following theorem shows that for infinite homogeneous portfolios the vector $\mu^{(m)} = (\mu_1^{(m)}, \ldots, \mu_m^{(m)})$ computed in (29) is optimal.

Theorem 2 *Let $\mu^{(1)} \in \mathbb{R}$ be a minimum of $\sigma^2_{\mathrm{ES}_\alpha(L_1^\infty)}$ and $\mu^{(m)} = (\mu_1^{(m)}, \ldots, \mu_m^{(m)})$ be defined by (29). Then*

$$\sigma^2_{\mathrm{ES}_\alpha(L_m^\infty)}(\mu^{(m)}) = \min\{\sigma^2_{\mathrm{ES}_\alpha(L_m^\infty)}(M) \mid M \in \mathbb{R}^m\}.$$

The proof of this theorem is based on the following proposition. It provides a general technique for reducing the minimization problem for a specific class of multivariate integrals to the minimization of one-dimensional integrals. In this proposition the variance and standard deviation of a random variable $U : \mathbb{R}^m \to \mathbb{R}$ with respect to $N_{\mathbf{0},C}$ are denoted by $\sigma^2(U)$ and $\sigma(U)$ respectively. C^{-1} and C^T are the inverse and transpose of the matrix C.

Proposition 2 *Let $A : \mathbb{R} \to \mathbb{R}_+$ be a real-valued, non-negative function and $M = (M_1, \ldots, M_m) \in \mathbb{R}^m$. Define $\mu \in \mathbb{R}$ by*

$$\mu := \frac{\mathrm{Cov}(U,V)}{\sigma(U)} = \frac{\mathrm{Cov}(U,V)}{\sqrt{R^2}}, \tag{31}$$

where $U : \mathbb{R}^m \to \mathbb{R}$ and $V : \mathbb{R}^m \to \mathbb{R}$ are the random variables

$$U(x_1, \ldots, x_m) := \sum_{j=1}^m \rho_j \cdot x_j, \qquad V(x_1, \ldots, x_m) := \sum_{i,j=1}^m x_i \cdot M_j \cdot C_{ij}^{-1}.$$

Then

$$\int A\left(\sum_{j=1}^m \rho_j \cdot x_j\right) \cdot \frac{n_{\mathbf{0},C}(x_1, \ldots, x_m)}{n_{M,C}(x_1, \ldots, x_m)}\, dN_{\mathbf{0},C} \geq \int A\left(\sqrt{R^2} \cdot x\right) \cdot \frac{n_{0,1}(x)}{n_{\mu,1}(x)}\, dN_{0,1}. \tag{32}$$

Equality holds in (32) if and only if $A = 0$ a.s. or M and μ satisfy the additional equation

$$\sigma^2(V) = \mu^2. \tag{33}$$

Proof: Since C^{-1} is symmetric, V^2 can be written in the form

$$V^2 = (M^T C^{-1} x)(x^T C^{-1} M)$$

and therefore

$$\sigma^2(V) = M^T C^{-1} C C^{-1} M = \sum_{i,j=1}^{m} M_i \cdot M_j \cdot C_{ij}^{-1}.$$

Hence,

$$\begin{aligned} \frac{n_{\mathbf{0},C}(x_1,\ldots,x_m)}{n_{M,C}(x_1,\ldots,x_m)} &= e^{(1/2)(\sum_{i,j=1}^{m}(x_i-M_i)(x_j-M_j)C_{ij}^{-1} - \sum_{i,j=1}^{m} x_i x_j C_{ij}^{-1})} \\ &= e^{(1/2)\sigma^2(V)-V}. \end{aligned} \tag{34}$$

Note that

$$(U(x_1,\ldots,x_m), V(x_1,\ldots,x_m)) \quad \text{and} \quad (\sqrt{R^2}\cdot x_1, \mu\cdot x_1 + \sqrt{\sigma^2(V)-\mu^2}\cdot x_2)$$

are both 2-dimensional Gaussian variables with the same joint distribution if considered as random variables on $(\mathbb{R}^m, N_{\mathbf{0},C})$ and $(\mathbb{R}^2, N_{\mathbf{0},I})$ respectively, where I denotes the identity matrix in $\mathbb{R}^2_2$. It follows from (34) and the independence of x_1 and x_2 on $(\mathbb{R}^2, N_{\mathbf{0},I})$ that

$$\begin{aligned} &\int A(\sum_{j=1}^{m}\rho_j\cdot x_j)\cdot \frac{n_{\mathbf{0},C}(x_1,\ldots,x_m)}{n_{M,C}(x_1,\ldots,x_m)}\, dN_{\mathbf{0},C} \;= \\ &\int A(U(x_1,\ldots,x_m))\cdot e^{(1/2)\sigma^2(V)-V(x_1,\ldots,x_m)}\, dN_{\mathbf{0},C} \;= \\ &\int A(\sqrt{R^2}\cdot x_1)\cdot e^{(1/2)\sigma^2(V)-(\mu\cdot x_1+\sqrt{\sigma^2(V)-\mu^2}\cdot x_2)}\, dN_{\mathbf{0},I} \;= \\ &B\cdot \int A(\sqrt{R^2}\cdot x)\cdot e^{(1/2)\mu^2-\mu\cdot x}\, dN_{0,1} \;= \\ &B\cdot \int A(\sqrt{R^2}\cdot x)\cdot \frac{n_{0,1}(x)}{n_{\mu,1}(x)}\, dN_{0,1}, \end{aligned} \tag{35}$$

where

$$B \;:=\; \int e^{(1/2)\sigma^2(V)-\sqrt{\sigma^2(V)-\mu^2}\cdot x-(1/2)\mu^2}\, dN_{0,1} \;=\; e^{\sigma^2(V)-\mu^2}.$$

It follows from the Cauchy-Schwarz inequality

$$\mathrm{Cov}(U,V)^2 \le \sigma^2(U)\cdot\sigma^2(V)$$

that $\mu^2 \leq \sigma^2(V)$ and therefore $B \geq 1$. Hence,

$$\int A(\sum_{j=1}^{m} \rho_j \cdot x_j) \cdot \frac{n_{\mathbf{0},C}(x_1, \ldots, x_m)}{n_{M,C}(x_1, \ldots, x_m)} \, dN_{\mathbf{0},C} \geq \int A(\sqrt{R^2} \cdot x) \cdot \frac{n_{0,1}(x)}{n_{\mu,1}(x)} \, dN_{0,1}.$$

Equality holds if and only if $A = 0$ a.s. or $B = 1$ which is equivalent to $\sigma^2(V) = \mu^2$. $\square$

Proof of Theorem 2: Define the real-valued, non-negative function $A : \mathbb{R} \to \mathbb{R}_+$ by

$$A(x) := (L^\infty(x) \cdot \mathbf{1}_{\{L^\infty(x) > c^\infty\}})^2.$$

Note that it follows from (5) and the definition (29) of $\mu^{(m)} = (\mu_1^{(m)}, \ldots, \mu_m^{(m)})$ that $\mu^{(m)}$ and $\mu^{(1)}$ satisfy equations (31) and (33). Let $M = (M_1, \ldots, M_m) \in \mathbb{R}^m$ and define $\mu \in \mathbb{R}$ such that (31) is satisfied. By Proposition 2 and the definition of $\mu^{(1)}$,

$$\begin{aligned}
\int \left(L_m^\infty \cdot \mathbf{1}_{\{L_m^\infty > c^\infty\}} \cdot \frac{n_{\mathbf{0},C}}{n_{\mu^{(m)},C}} \right)^2 dN_{\mu^{(m)},C} &= \int A(\sum_{j=1}^{m} \rho_j \cdot x_j) \cdot \frac{n_{\mathbf{0},C}}{n_{\mu^{(m)},C}} \, dN_{\mathbf{0},C} \\
&= \int A(\sqrt{R^2} \cdot x) \cdot \frac{n_{0,1}(x)}{n_{\mu^{(1)},1}(x)} \, dN_{0,1} \\
&\leq \int A(\sqrt{R^2} \cdot x) \cdot \frac{n_{0,1}(x)}{n_{\mu,1}(x)} \, dN_{0,1} \\
&\leq \int A(\sum_{j=1}^{m} \rho_j \cdot x_j) \cdot \frac{n_{\mathbf{0},C}}{n_{M,C}} \, dN_{\mathbf{0},C} \\
&= \int \left(L_m^\infty \cdot \mathbf{1}_{\{L_m^\infty > c^\infty\}} \cdot \frac{n_{\mathbf{0},C}}{n_{M,C}} \right)^2 dN_{M,C}.
\end{aligned}$$

Together with the representation (28) of $\sigma^2_{\mathrm{ES}_\alpha(L_m^\infty)}(M)$, this proves the theorem. $\square$

4.5 Numerical analysis

Importance sampling based on the shift vector $(\mu_1^{(m)}, \ldots, \mu_m^{(m)})$ in (29) minimizes the Monte Carlo sampling fluctuation for infinitely homogeneous portfolios. We do not know yet, however, whether this technique leads to a significant error reduction in the Monte Carlo based estimation of expected shortfall $\mathrm{ES}_\alpha(L)$ and expected shortfall contributions $\mathrm{ESC}_\alpha(L_i, L)$ for realistic portfolios. In order to assess its impact on the portfolio risk measure expected shortfall $\mathrm{ES}_\alpha(L)$ we apply importance sampling to a large loan portfolio and calculate the standard deviation of the Monte Carlo estimator for $\mathrm{ES}_\alpha(L)$ with $\alpha = 99.9\%$.

The test portfolio consists of 25000 loans with an inhomogeneous exposure and default probability distribution. The average exposure size is 0.004% of the total exposure and the standard deviation of the exposure size is 0.026%. Default probabilities vary between 0.02% and 27%. The portfolio expected loss is 0.72% and the unexpected loss, i.e. the standard deviation, is 0.87%. Default correlations are specified by the KMV factor model (see Kealhofer and Bohn (2001) for a description of the model), comprising 96 systematic country and industry factors. Although the portfolio is relatively well diversified there are concentrations caused by exposures to a single sector (geographic region or industry) or to several highly correlated sectors. Name concentrations do not play a dominant role. The test portfolio is a typical example of a large credit portfolio in an international bank. We expect that the variance reductions reported in this paper can be reproduced with any portfolio of similar characteristics.

In Figure 1 we plot the standard deviation of the Monte Carlo estimator for $\mathrm{ES}_{0.999}(L)$ as a function of the norm of the vector $(\mu_1^{(m)}, \ldots, \mu_m^{(m)})$. A scaling factor of 0 corresponds to no importance sampling, whereas a scaling factor of 1 corresponds to $(\mu_1^{(m)}, \ldots, \mu_m^{(m)})$. The other points represent vectors with identical direction but a linearly interpolated/extrapolated norm, i.e. 0.5 corresponds to the vector $0.5 \cdot (\mu_1^{(m)}, \ldots, \mu_m^{(m)})$. These results were obtained from a sample of 40 independent Monte Carlo runs of 10000 simulation trials for each scaling factor. On the right-hand axis we plot the average result from the 40 runs with the standard deviation in error bars.

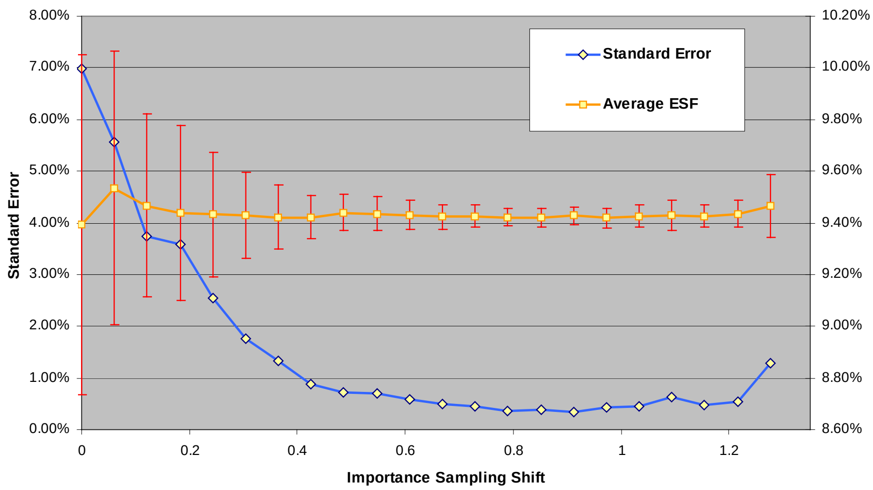

Figure 1: Monte Carlo sampling error as a function of the importance sampling shift.

From these results we draw two conclusions. First we have demonstrated that importance sampling can significantly improve the quality of the Monte Carlo estimate of the expected shortfall measure. The variance ratio between the optimal point in the graph and the no-shift case is 400, i.e. the same precision without any importance sampling would require 400 times more simulations. Improvements of a comparable magnitude were found for the Monte Carlo estimate of the quantile of the loss distribution, i.e. the value-at-risk.

Secondly, we observe that our theoretical optimal shift size slightly overestimates the empirical optimal shift. Our explanation for this is that in our determination of the equivalent homogeneous portfolio we have overestimated the average correlation R^2. This has been confirmed by the observation that the R^2 determined from fitting a Vasicek distribution to our best Monte Carlo estimate for the 99.9% quantile is approximately 10% smaller than the one calculated from the approximation procedure in Section 2.2.

Compared to the risk measure expected shortfall, the calculation of numerically stable expected shortfall contributions of individual loans is an even more challenging task. Importance sampling on systematic factors also leads to a significant reduction in the volatility of the $\mathrm{ESC}_\alpha(L_i, L)$: in the test calculations in Section 6, the variance is reduced by a factor of more than 100 (see Table 2), for 75% of the loans the standard deviation is below 5% if 100000 simulations are calculated (see Figure 3). However, for a number of transactions in the test portfolio the statistical fluctuations of their expected shortfall contributions are still unacceptably high. In the following section we obtain further improvements by utilizing the independence of the loan loss variables $L_1, \ldots, L_n$ conditional on given values $\chi = (\chi_1, \ldots, \chi_m)$ of the systematic variables $x = (x_1, \ldots, x_m)$.

5 Variance reduction based on conditional independence of specific factors

There exist several options how to exploit conditional independence for stabilizing expected shortfall contributions. In this section, we deal with three different techniques:

1. importance sampling of specific factors based on exponential twisting of default probabilities,
2. deterministic calculation of the expected shortfall contribution of the i-th loan in scenarios where values of all systematic factors and all but the i-th specific factor have been simulated and
3. analytic approximations of conditional loss distributions motivated by the application of the central limit theorem.

Numerical results and comparisons are presented in Section 6.

5.1 Importance sampling on specific factors

Conditional on $\{x = \chi\}$, Glasserman and Li (2005) and Merino and Nyfeler (2004) suggest importance sampling based on exponential twists of default probabilities to stabilize expected shortfall. More precisely, they consider the related problem of improving the MC estimate of the tail probability $P(L > c)$.

It is intuitively clear that better estimates of $P(L > c)$ can be obtained by increasing the conditional default probabilities of the individual loans, i.e by replacing each conditional default probability

$$p_i(\chi) = N\left(\frac{N^{-1}(p_i) - \sum_{j=1}^{m} \phi_{ij}\chi_j}{\sqrt{1 - R_i^2}}\right), \qquad i = 1, \ldots, n, \tag{36}$$

by a higher default probability $\bar{p}_i(\chi)$. Glasserman and Li (2005) prove asymptotic optimality for the exponential twist

$$\bar{p}_i(\chi) := \frac{p_i(\chi)e^{\theta_c(\chi)\, l_i}}{1 + p_i(\chi)(e^{\theta_c(\chi)\, l_i} - 1)}, \tag{37}$$

where $\theta_c(\chi)$ denotes the saddle-point defined in Section 2.2.2. Hence, the basic importance sampling identity becomes

$$P(L > c) = E_{\bar{p}}\left(\mathbf{1}_{\{L>c\}} \prod_{i=1}^{n} \left(\frac{p_i(\chi)}{\bar{p}_i(\chi)}\right)^{A_i(\chi,z_i)} \left(\frac{1 - p_i(\chi)}{1 - \bar{p}_i(\chi)}\right)^{1-A_i(\chi,z_i)}\right),$$

where $E_{\bar{p}}$ denotes the expectation using the new default probabilities $\bar{p}_1(\chi), \ldots, \bar{p}_n(\chi)$ and $A_i(\chi, z_i)$ is the i-th Ability-to-Pay variable restricted to $\{x = \chi\}$.[8]

It is straightforward to combine the importance sampling techniques on systematic and specific factors:

1. Apply importance sampling to the systematic factors $x_1, \ldots, x_m$ and compute samples
$$\chi_1 = (\chi_{11}, \ldots, \chi_{1m}), \ldots, \chi_k = (\chi_{k1}, \ldots, \chi_{km}).$$
2. For each of the k systematic samples $\chi_j = (\chi_{j1}, \ldots, \chi_{jm})$: calculate l IS samples of the specific factors $z_1, \ldots, z_n$ using the default probabilities $\bar{p}_i(\chi_j)$ in (37).

The relative importance of both IS techniques depends on the characteristics of the portfolio, in particular on the degree of correlation (Glasserman and Li, 2005). In our setting, the variance reduction on the systematic factors clearly dominates for the 25000 loan portfolio defined in Section 4.5. Numerical results are presented in Section 6.

[8]In Glasserman (2005), exponential twisting is used to derive an asymptotic approximation to conditional default probabilities.

5.2 Conditional expected shortfall allocation

The variance reduction technique presented in this subsection can be combined with importance sampling on systematic and specific factors. It utilizes the simple form of $\mathrm{E}(L_i \,|\, L > c)$, $i = 1, \dots, n$, conditional on given values of the systematic variables $x_1, \dots, x_m$ and the remaining specific variables $z_1, \dots, z_{i-1}, z_{i+1} \dots, z_n$ (compare to Merino and Nyfeler (2004)).

Let $(\chi, \sigma) = (\chi_1, \dots, \chi_m, \sigma_1, \dots, \sigma_n) \in \mathbb{R}^{m+n}$, $i \in \{1, \dots, n\}$. Our objective is to calculate $\mathrm{E}(L_i \,|\, L > c)$ on $\Omega_i(\chi, \sigma)$, where $\Omega_i(\chi, \sigma)$ denotes the subset

$$\{x_1 = \chi_1, \dots, x_m = \chi_m, z_1 = \sigma_1, \dots, z_{i-1} = \sigma_{i-1}, z_{i+1} = \sigma_{i+1}, \dots, z_n = \sigma_n\}$$

of $\mathbb{R}^{m+n}$.

Since $\sum_{j \neq i} L_j$ is deterministic on $\Omega_i(\chi, \sigma)$ we distinguish three cases:

$$\begin{aligned}
\sum_{j \neq i} L_j > c : &\quad \mathrm{E}(L_i \mathbf{1}_{\{L>c\}} \,|\, \Omega_i(\chi, \sigma)) = p_i(\chi) l_i, \quad \mathbb{P}(L > c \,|\, \Omega_i(\chi, \sigma)) = 1, \\
c \geq \sum_{j \neq i} L_j > c - l_i : &\quad \mathrm{E}(L_i \mathbf{1}_{\{L>c\}} \,|\, \Omega_i(\chi, \sigma)) = p_i(\chi) l_i, \quad \mathbb{P}(L > c \,|\, \Omega_i(\chi, \sigma)) = p_i(\chi), \\
c - l_i \geq \sum_{j \neq i} L_j : &\quad \mathrm{E}(L_i \mathbf{1}_{\{L>c\}} \,|\, \Omega_i(\chi, \sigma)) = 0, \qquad \mathbb{P}(L > c | \Omega_i(\chi, \sigma)) = 0,
\end{aligned}$$

where the conditional default probability $p_i(\chi)$ of the i-th loan on $\Omega_i(\chi, \sigma)$ is specified in (36).

These simple formulae can be combined with importance sampling in order to improve the stability of expected shortfall allocation. Let

$$(\chi_j, \sigma_j) = (\chi_{j1}, \dots, \chi_{jm}, \sigma_{j1}, \dots, \sigma_{jn}), \quad j = 1, \dots, k$$

be k samples of the systematic factors $x_1, \dots, x_m$ and the specific factors $z_1, \dots, z_n$ and denote the probabilities of the samples by $q(\chi_1, \sigma_1), \dots, q(\chi_k, \sigma_k)$. The expected shortfall contribution of the i-th loan equals

$$\begin{aligned}
\mathrm{E}(L_i \,|\, L > c) &= \frac{\mathrm{E}(L_i \mathbf{1}_{\{L>c\}})}{\mathbb{P}(L > c)} \\
&\approx \frac{\sum_{j=1}^{k} q(\chi_j, \sigma_j) \cdot \mathrm{E}(L_i \mathbf{1}_{\{L>c\}} \,|\, \Omega_i(\chi_j, \sigma_j))}{\sum_{j=1}^{k} q(\chi_j, \sigma_j) \cdot \mathbb{P}(L > c \,|\, \Omega_i(\chi_j, \sigma_j))}, \qquad (38)
\end{aligned}$$

where (38) can be easily obtained from the above formulae.

The variance reduction in (38) is due to the fact that the simulation of the i-th specific factor has been replaced by a deterministic calculation of $\mathrm{E}(L_i \,|\, L > c)$ in each scenario $\Omega_i(\chi_j, \sigma_j)$. This simple technique is easy to implement and does not require much additional computing time. The variance reduction obtained in our test portfolio is significant, particularly in combination with importance sampling on the systematic factors (see Section 6). Furthermore, this technique can be generalized to migration mode in a straightforward way (Section 7).

5.3 Normal approximations

5.3.1 Calculation of expected shortfall for Gaussian distributions

Conditional on a systematic scenario $\{x = \chi\}$, the volatility of the expected shortfall estimates $\mathrm{E}(L_i \,|\, L > c)$ can be completely eliminated if the conditional portfolio loss is not simulated but approximated by an analytic distribution. The analytic approximation of the portfolio loss L in Theorem 1 has been obtained by applying the law of large numbers to the sum of the independent loss variables conditional on $x = \chi$. A more precise approximation of L is based on the central limit theorem: approximate $L = \sum_{i=1}^{n} L_i$ on $\{x = \chi\}$ by a normal distribution $L(\chi)$ with mean and variance

$$\mu(\chi) := \sum_{i=1}^{n} l_i \cdot p_i(\chi), \qquad \sigma^2(\chi) := \sum_{i=1}^{n} l_i^2 \cdot p_i(\chi) \cdot (1 - p_i(\chi)). \tag{39}$$

We will now apply this technique to obtain an approximation of $\mathrm{E}(L_i \, \mathbf{1}_{\{L>c\}})$ in a systematic scenario $\{x = \chi\}$. Firstly, $\{x = \chi\}$ is split into two components: $\{x = \chi\} \cap \{A_i \leq D_i\}$ and $\{x = \chi\} \cap \{A_i > D_i\}$. By the central limit theorem, L can be approximated by a normal distribution $L_i(\chi)$ on $\{x = \chi\} \cap \{A_i \leq D_i\}$, where the mean and variance of $L_i(\chi)$ are adjusted to

$$\mu_i(\chi) := \sum_{j=1,\ j\neq i}^{n} l_j \cdot p_j(\chi) + l_i, \qquad \sigma_i^2(\chi) = \sum_{j=1,\ j\neq i}^{n} l_j^2 \cdot p_j(\chi) \cdot (1 - p_j(\chi)).$$

Hence, on $\{x = \chi\} \cap \{A_i \leq D_i\}$,

$$\begin{aligned} \mathbb{P}(L > c \,|\, \{x = \chi\} \cap \{A_i \leq D_i\}) &\approx \mathbb{P}(L_i(\chi) > c \,|\, \{x = \chi\} \cap \{A_i \leq D_i\}) \\ &= 1 - N_{\mu_i(\chi), \sigma_i^2(\chi)}(c). \end{aligned} \tag{40}$$

Since $L_i = 0$ on $\{A_i > D_i\}$ and $L_i = l_i$ on $\{A_i \leq D_i\}$, the following approximation of expected shortfall contributions on $\{x = \chi\}$ is derived from (40):

$$\begin{aligned} \mathrm{E}(L_i \, \mathbf{1}_{\{L>c\}} \,|\, \{x = \chi\}) &\approx p_i(\chi) \cdot \mathrm{E}(L_i \, \mathbf{1}_{\{L_i(\chi)>c\}} \,|\, \{x = \chi\} \cap \{A_i \leq D_i\}) \\ &= p_i(\chi) \cdot l_i \cdot \left(1 - N_{\mu_i(\chi), \sigma_i^2(\chi)}(c)\right). \end{aligned} \tag{41}$$

5.3.2 Combining importance sampling and analytic approximations

The following algorithm calculates expected shortfall contributions for a given confidence level $\alpha \in (0, 1)$. It uses importance sampling for the simulation of the systematic factors and applies normal approximations to the conditional loss distributions.

Algorithm:

1. *Importance sampling on systematic factors:*

   Compute k samples

   $$\chi_1 = (\chi_{11}, \ldots, \chi_{1m}), \ldots, \chi_k = (\chi_{k1}, \ldots, \chi_{km})$$

   of the systematic factors $x_1, \ldots, x_m$ using the importance sampling technique presented in Section 4. Denote the probabilities of the samples by $q(\chi_1), \ldots, q(\chi_k)$.

2. *Normal approximations of conditional distributions:*

   For each $j \in \{1, \ldots, k\}$: compute the mean $\mu(\chi_j)$ and the variance $\sigma^2(\chi_j)$ of the normal approximation $L(\chi_j)$ on $\{x = \chi_j\}$ (according to definition (39)). These normal distributions define a loss distribution on the space

   $$\bigcup_{j=1}^{k} \{x = \chi_j\}$$

   which approximates the portfolio loss distribution L.

3. *Approximation of* $c = \mathrm{VaR}_\alpha(L)$:

   An estimate of $c = \mathrm{VaR}_\alpha(L)$ can be backed out of the estimation

   $$\alpha \approx \sum_{j=1}^{k} q(\chi_j)\ N_{\mu(\chi_j),\sigma^2(\chi_j)}(c).$$

4. *Calculation of the expected shortfall contributions:*

   For each $i \in \{1, \ldots, n\}$, the expected shortfall contribution $\mathrm{ESC}_\alpha(L_i, L)$ is now approximated by

   $$\mathrm{ESC}_\alpha(L_i, L) \approx \frac{\sum_{j=1}^{k} q(\chi_j)\ \mathrm{E}(L_i\, \mathbf{1}_{\{L>c\}} \,|\, \{x = \chi_j\})}{1-\alpha}\ ,$$

   where $\mathrm{E}(L_i\, \mathbf{1}_{\{L>c\}} \,|\, \{x = \chi_j\})$ is calculated by (41), i.e. it is derived from the normal approximation $L_i(\chi_j)$ on $\{x = \chi_j\} \cap \{A_i \leq D_i\}$.

Analytic approximations inevitably introduce errors into the calculation of the capital estimates. The accuracy of the portfolio approximation and the estimates for $\mathrm{ESC}_\alpha(L_i, L)$ clearly depends on the characteristics of the portfolio, in particular the homogeneity of the exposures. Since the typical credit portfolio of a large international bank is rather well-diversified we experienced relatively small errors in the calculation of $\mathrm{ESC}_\alpha(L_i, L)$ (see Tables 1 and 2 in Section 6). However, it is certainly a worthwhile topic for future research to develop analytic approximation techniques that are applicable to large credit portfolios and provide a better fit to the conditional distribution of L in a systematic scenario as well as efficient formulae for the calculation of ES contributions.

6 Numerical results

We now apply the different variance reduction techniques presented in this paper to the test portfolio specified in Section 4.5, as well as to a smaller portfolio of 1000 loans. The objective is to compare these techniques in terms of the numerical stability and accuracy of the results that they produce, and hence assess their suitability for allocating economic capital to individual transactions.

A common feature of the analyzed algorithms is the split of the calculation of expected shortfall contributions into two steps:

1. simulation of systematic factors,

2. calculation of expected shortfall contributions by utilizing the independence of loss variables in each systematic scenario.

The simulation of the systematic factors is either based on straightforward Monte Carlo simulation (MC) or on the importance sampling technique (IS) developed in Section 4. In each systematic scenario, the following techniques, presented in Section 5, are applied to the specific factors:

1. straightforward Monte Carlo simulation (MC),

2. Monte Carlo simulation with importance sampling based on exponential twisting (IS),

3. Monte Carlo simulation with the conditional allocation (CA),

4. approximation of the conditional loss distribution by a normal distribution, i.e. application of the central limit theorem (NA).

The calculations are based on 20 runs with 100000 MC samples, i.e. 100000 simulations of the systematic factors and one simulation of all specific factors (or a normal approximation of the conditional loss distribution) in each systematic scenario. In order to assess the accuracy of the different techniques we compare the average ES contribution (calculated as the mean of 20 estimates) of a loan to a benchmark. The benchmark value is based on the average of 28 runs, where each run uses 1000000 simulations with importance sampling on the systematic factors and straightforward MC simulation on the specific factors.

Firstly we analyze the different methods as applied to the 25000 loan portfolio used previously. For each loan, the difference between its average ES contribution and the benchmark is calculated and expressed in % of benchmark value. In Table 1, the mean of these 25000 relative differences is exhibited for the 8 calculation methods.

Next, we calculate the standard deviation of the simulated ES contributions. More precisely, for each loan the standard deviation of the 20 estimates of its ES

% Diff		Specific Factors			
		MC	IS	CA	NA
Systematic Factors	MC	9.8	9.0	2.6	2.4
	IS	0.8	0.7	0.1	1.7

Table 1: Average relative error in 25000 loan portfolio.

contributions is calculated. The second table shows the average standard deviation again expressed in % of benchmark value.

% StDev		Specific Factors			
		MC	IS	CA	NA
Systematic Factors	MC	56.6	53.9	12.3	6.1
	IS	4.6	3.7	0.9	0.5

Table 2: Average standard deviation in 25000 loan portfolio.

A comparison of the different variance reduction techniques shows that the impact of IS on systematic factors is most significant. This is due to the characteristics of the portfolio: our test portfolio is large and granular, i.e. it is not dominated by individual names. As a consequence, the homogeneous portfolio approximation used in Section 4.3 as basis for IS on systematic factors, provides a good representation. Although the portfolio is relatively well diversified there are concentrations caused by exposures to a single sector (geographic region or industry) or to several highly correlated sectors. These concentrations are exploited by importance sampling on systematic factors.

Because of the portfolio characteristics it is not surprising that combining IS on systematic factors with IS on specific factors only provides a small additional reduction of the variance: the portfolio loss depends on concentration risks rather than on the behaviour of individual loans, and the exponential twist does not have the granularity to closely fit the requirements of such a large number of diverse loans.

We have observed a better performance of importance sampling on specific factors for smaller portfolios with low dependence on systematic factors (compare to Glasserman and Li (2005)). As an example, we analyse the stability and accuracy of the results on a smaller portfolio, made up of 1000 loans of our original portfolio. Then importance sampling applied to the specific factors improves the stability and accuracy of results more than that applied to the systematic factors. Indeed, as we move to the smaller 1000 loan portfolio, the influence of the systematic factors on portfolio loss diminishes whilst that of the specific factors increases. Furthermore, the homogenous portfolio gives a less accurate representation of the smaller portfolio, and the exponential twist is able to fit more closely to an optimal solution for 1000 loans rather than 25000 (see Tables 3 and 4).

% Diff		Specific Factors			
		MC	IS	CA	NA
Systematic Factors	MC	13.2	5.2	4.0	75.1
	IS	5.6	1.9	2.1	79.4

Table 3: Average relative error in 1000 loan portfolio.

% StDev		Specific Factors			
		MC	IS	CA	NA
Systematic Factors	MC	75.6	26.3	17.9	20.3
	IS	29.3	3.8	8.4	0.7

Table 4: Average standard deviation in 1000 loan portfolio.

An additional problem of the calculation of exponentially twisted default probabilities is the long computing time if this technique is applied in a large number of systematic scenarios. We have therefore repeated the calculation with only 10000 systematic scenarios but 10 specific scenarios for each systematic sample: as expected, the computing time is reduced but the volatility of the expected shortfall contributions is higher than in the original setup.

Turning now to the other methods of calculation, the combination of importance sampling on systematic factors with normal approximations of conditional loss distributions is the most stable method. This is not surprising because the simulation of the specific factors is replaced by a purely deterministic technique. The main disadvantage of the normally distributed approximations is the deviation of the estimates from the correct expected shortfall contributions. Even for the large 25000 loan portfolio these differences are relevant if expected shortfall allocation is calculated at high quantiles like the 99.9% quantile used in this analysis. Furthermore, the smaller the portfolio, the more inaccurate is the normal approximation of the conditional loss distributions, so that the results observed on the 1000 loan portfolio disqualify the method altogether.

Despite its simplicity the conditional allocation performed well in our analysis on the large and acceptably on the small portfolios: if combined with IS on systematic factors the difference to the benchmark is 0.1% on average and the standard deviation is 0.9% on the 25000 loan portfolio, and 2.1% and 8.4% on the 1000 loan portfolio. Whilst other methods are tied to the portfolio size, with importance sampling on systematic factors and the normal approximation being more suited to large portfolios, and importance sampling on the specific factors more suited to small, the relative improvement to the stability and accuracy of results under the conditional method remains good irrespective of portfolio size.

Tables 1 and 2 provide information on the average error and the average volatility of the MC estimates in the 25000 loan portfolio. A more detailed analysis of the estimated ES contributions for individual loans can be found in Figures 2 and 3. For each of the 25000 loans we have estimated

1. the relative difference between its average ES contribution and the benchmark,

2. the standard deviation of its ES contributions

from 20 independent runs of 100000 simulation trials

1. with straightforward Monte Carlo simulation (MC),

2. with importance sampling on the systematic factors (IS),

3. with importance sampling on the systematic factors and conditional allocation (IS + Cond).

On the horizontal axis in Figure 2 we divide the relative difference to the benchmark in bins and on the vertical axis we display the number of loans in each bin. Note that the bins on the horizontal axis are not of equal size.

In Figure 3, the standard deviations of the 25000 ES contributions are displayed in the same way.

The improvements obtained by using importance sampling on systematic factors and conditional allocation are significant: the standard deviation is less than 1% for 70% of the loans, more than 99% have a standard deviation below 4%. It would require $4000 \approx (56.6\%/0.9\%)^2$ times more simulations to achieve a similar precision with straightforward Monte Carlo simulation. Another advantage of this variance reduction technique is that it can be easily generalized to multi-state models. This extension will be discussed in the next section.

Using IS on systematic factors and conditional allocation increases the computing time compared to straightforward Monte Carlo simulation. However, the increase is rather modest compared to the variance reduction achieved: in our implementation we observed a factor of 2.

We conclude that IS on systematic factors together with conditional allocation is a very efficient technique for stabilizing ES contributions in large portfolios such as those used to model the credit risk of a financial institution. For smaller portfolios, a combination of IS on systematic and specific factors (potentially in conjunction with conditional allocation) seems to be more suitable.

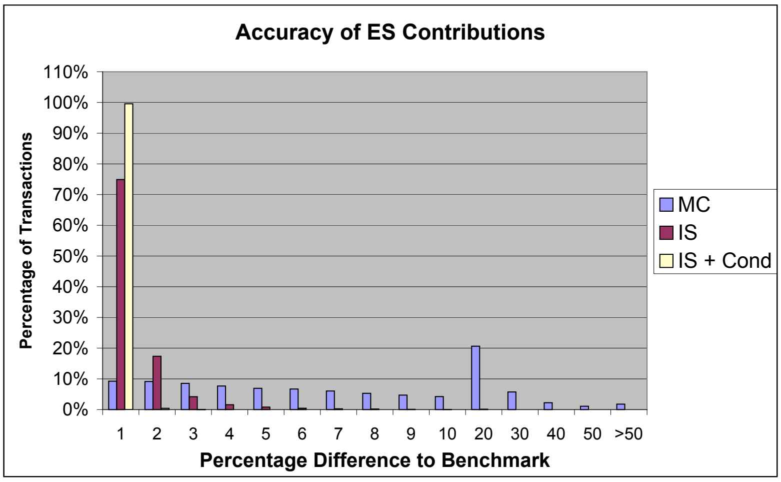

Figure 2: Distribution of the relative difference of ES contributions to the benchmark.

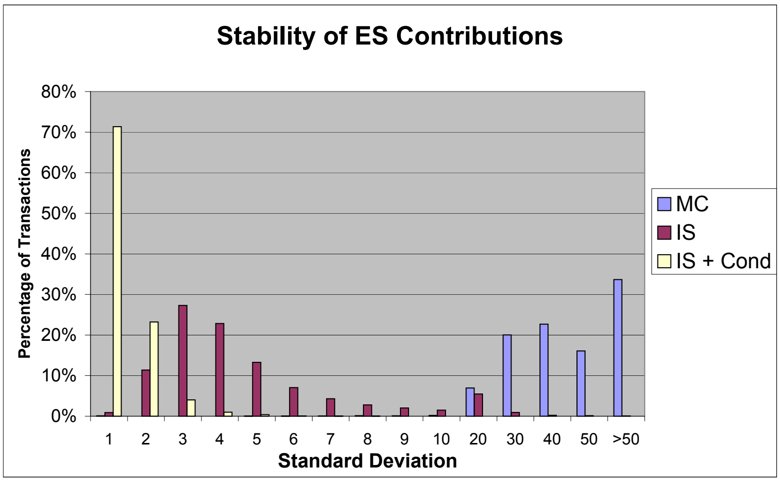

Figure 3: Distribution of statistical fluctuation of ES contributions.

7 Generalization to multi-state models

The portfolio model defined in Section 2.1 only distinguishes between two states at horizon: default and non-default. The objective of this section is the introduction of rating migration and the generalization of importance sampling on systematic factors and conditional allocation to the multi-state rating model.

7.1 Rating migration

Let r be the number of rating classes including default and define thresholds

$$-\infty = D_i^{(0)} \leq D_i^{(1)} \leq \ldots \leq D_i^{(r-1)} \leq D_i^{(r)} = \infty, \qquad i = 1, \ldots, n$$

for each of the n loans. The event $\{D_i^{(k-1)} < A_i \leq D_i^{(k)}\}$ is interpreted as the i-th loan is in rating class k at horizon. The migration probabilities $p_i^{(k)} := \mathbb{P}(\{D_i^{(k-1)} < A_i \leq D_i^{(k)}\})$ and migration thresholds $D_i^{(k)}$ are usually derived from a rating migration matrix. Note that the default probability p_i equals $p_i^{(1)} = \mathbb{P}(\{A_i \leq D_i^{(1)}\})$.

The loss $l_i^{(k)}$ of the i-th loan in migration class k is specified by a vector

$$(l_i^{(1)}, \ldots, l_i^{(r)}) \in \mathbb{R}^r \text{ with } l_i^{(1)} \geq \ldots \geq l_i^{(r)}.$$

The loan loss $L_i : \mathbb{R}^{m+1} \to \mathbb{R}$ is generalized to

$$L_i := \sum_{k=1}^{r} l_i^{(k)} \cdot \mathbf{1}_{\{D_i^{(k-1)} < A_i \leq D_i^{(k)}\}}.$$

7.2 Importance sampling in a multi-state model

The first step in adapting importance sampling to the multi-state model is the construction of a homogeneous portfolio. The definitions of the homogeneous loss vector $(l^{(1)}, \ldots, l^{(r)})$ and the homogeneous migration probabilities $p^{(1)}, \ldots, p^{(r)}$ generalize (3):

$$l^{(k)} := \frac{\sum_{i=1}^{n} l_i^{(k)}}{n}, \qquad p^{(k)} := \frac{\sum_{i=1}^{n} p_i^{(k)} l_i^{(1)}}{\sum_{i=1}^{n} l_i^{(1)}}, \qquad k = 1, \ldots, r.$$

In line with the two-state model, the positive weights $g_1, \ldots, g_n \in \mathbb{R}$ are given by $g_i := \mathrm{E}(L_i) = \sum_{k=1}^{r} p_i^{(k)} l_i^{(k)}$ and the homogeneous weight vector $\rho = (\rho_1, \ldots, \rho_m)$ and the homogeneous R^2 are defined by (4), (5) and (6).

In the second step, the computation of the optimal drift vector $\mu^{(m)}$ is modified. As in the two-state model, $\mu^{(m)}$ is derived from the minimum $\mu^{(1)}$ of

$$\int_{-\infty}^{N^{-1}(1-\alpha)} \frac{(L_1^{\infty} \cdot n_{0,1})^2}{n_{M,1}} \, dx.$$

However, a straightforward generalization of Theorem 1 shows that L_1^{∞} has a more complex form in the multi-state model: $L_1^{\infty}(x)$ equals

$$n \cdot \sum_{k=1}^{r} l^{(k)} \cdot \left(N\left(\frac{N^{-1}(\sum_{j=1}^{k} p^{(j)}) - \sqrt{R^2}x}{\sqrt{1-R^2}} \right) - N\left(\frac{N^{-1}(\sum_{j=1}^{k-1} p^{(j)}) - \sqrt{R^2}x}{\sqrt{1-R^2}} \right) \right).$$

Analogously to the two-state model, the application of formula (29) lifts the one-dimensional minimum $\mu^{(1)}$ to the m-dimensional vector $\mu^{(m)} = (\mu_1^{(m)}, \ldots, \mu_m^{(m)})$, i.e.

$$\mu_i^{(m)} := \frac{\mu^{(1)} \cdot \sum_{j=1}^m \mathrm{Cov}(x_i, x_j) \cdot \rho_j}{\sqrt{R^2}},$$

that specifies the shifted mean of the systematic factors used in the importance sampling algorithm.

7.3 Conditional allocation in a multi-state model

The conditional expected shortfall allocation can be easily generalized to the rating migration model. As in Section 5, $\Omega_i(\chi, \sigma)$ denotes the set

$$\{x_1 = \chi_1, \ldots, x_m = \chi_m, z_1 = \sigma_1, \ldots, z_{i-1} = \sigma_{i-1}, z_{i+1} = \sigma_{i+1}, \ldots, z_n = \sigma_n\}$$

for $i \in \{1, \ldots, n\}$ and $(\chi, \sigma) = (\chi_1, \ldots, \chi_m, \sigma_1, \ldots, \sigma_n) \in \mathbb{R}^{m+n}$. The conditional migration probabilities of the i-th loan on $\Omega_i(\chi, \sigma)$ equal

$$p_i^{(k)}(\chi) = N\left(\frac{D_i^{(k)} - \sum_{j=1}^m \phi_{ij}\chi_j}{\sqrt{1 - R_i^2}}\right) - N\left(\frac{D_i^{(k-1)} - \sum_{j=1}^m \phi_{ij}\chi_j}{\sqrt{1 - R_i^2}}\right) \qquad k = 1, \ldots, r.$$

Since $\sum_{j \neq i} L_j$ is deterministic on $\Omega_i(\chi, \sigma)$, we define

$$K_i := \begin{cases} \max\{k \in \{1, \ldots, r\} \mid \sum_{j \neq i} L_j > c - l_i^{(k)}\} & \text{if } \sum_{j \neq i} L_j > c - l_i^{(1)} \text{ on } \Omega_i(\chi, \sigma), \\ 0 & \text{if } \sum_{j \neq i} L_j \leq c - l_i^{(1)} \text{ on } \Omega_i(\chi, \sigma) \end{cases}$$

and obtain

$$\mathrm{E}(L_i \mathbf{1}_{\{L>c\}} \,|\, \Omega_i(\chi, \sigma)) = \sum_{k=1}^{K_i} p_i^{(k)}(\chi) \cdot l_i, \qquad \mathbb{P}(L > c \,|\, \Omega_i(\chi, \sigma)) = \sum_{k=1}^{K_i} p_i^{(k)}(\chi). \quad (42)$$

Importance sampling on systematic factors and conditional allocation on specific factors are combined in the multi-factor model analogously to the two-state mode. Using the generalized formulae (42), the calculation of $\mathrm{E}(L_i \,|\, L > c)$ immediately follows from (38).

8 Conclusion

In the framework of a standard structural credit portfolio model, we investigated the numerical estimation of capital allocation according to expected shortfall. As it is a coherent measure of risk and has a particularly intuitive interpretation, expected shortfall has long been viewed as a desirable basis for capital allocation. Its estimation however has proved to be almost intractable for large credit portfolios due to expected shortfall's focus on the extreme tail of the loss distribution.

We examined several variance reduction techniques for the Monte Carlo based estimation of expected shortfall contributions in large loan portfolios. Firstly, we gave a detailed presentation of an importance sampling technique developed at Deutsche Bank. It applies to the systematic factors of the portfolio model and is based on the infinite granularity approximation of the portfolio loss distribution. This method was used in conjunction with three further variance reduction techniques that utilize the independence of the specific factors of the model: importance sampling based on exponential twisting of conditional default probabilities, normal approximation of the portfolio loss distribution in a given systematic scenario, and the semi-analytical conditional approach.

We compared the performance of these methods in terms of the accuracy and Monte Carlo stability of their results by testing them on a large portfolio of 25000 loans as well as on a smaller portfolio of 1000 loans. We found that the combination of importance sampling techniques on systematic and specific factors performed well on the 1000 loan portfolio. Importance sampling on systematic factors in conjunction with conditional allocation proved to be an efficient technique for stabilizing expected shortfall allocation in larger credit portfolios: applied to our test portfolio of 25000 loans it reduces the variance of the Monte Carlo estimate of expected shortfall contributions of individual loans - and therefore the number of required simulations - by a factor of 4000. Additionally, this approach is not computationally demanding, and its simplicity makes it open to methodological extensions. The generalization to multi-state models given in this paper is particularly important for practical applications. In summary, our results show that the inherent numerical problems of expected shortfall allocation in structural credit portfolio models can be overcome and, as a consequence, economic capital allocation according to expected shortfall is a viable option for financial institutions.

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