Sensible and Efficient Capital Allocation for Credit Portfolios

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Abstract

The expected shortfall and volatility capital allocation schemes are analyzed from a formal mathematical perspective and in a simulation study. We argue that expected shortfall is the superior measure for the allocation of capital in credit portfolios. The simulation is based on a new importance sampling algorithm for Merton-type models, which greatly increases the precision of Monte-Carlo estimates.

1 Introduction

Portfolio modeling has two main objectives: the quantification of portfolio risk, which is usually expressed as the economic capital of the portfolio, and its allocation to subportfolios and individual transactions. The standard approach in credit portfolio modeling is to define the economic capital in terms of a quantile of the portfolio loss distribution. The capital charge of an individual transaction is usually based on a covariance technique and called volatility contribution.[1]

In this paper we present evidence that the combination of quantiles and covariances is not a satisfactory approach to risk measurement and capital allocation in credit portfolios. An alternative definition of economic capital is based on the expected shortfall, which can intuitively be interpreted as the average of all losses above a given quantile of the loss distribution. Moreover, there is a natural way to allocate the expected shortfall of the portfolio: the expected shortfall contribution of a transaction is its average contribution to the portfolio losses above the specified quantile.

Starting from a set of practical requirements for a capital allocation scheme we introduce an axiomatic framework to formalize these requirements and demonstrate that expected shortfall is a consistent capital allocation in this perspective. The main shortcomings of quantiles and volatility contributions, on the other hand, are the missing diversification property and the fact that the volatility contribution of a loan may exceed its exposure.

The main focus of our analysis is on the practical application of both allocation schemes to realistic credit portfolios. The most noticeable result is that the expected shortfall allocation detects concentration risk more accurately. Furthermore, we observe that volatility contributions typically overestimate the risk of poorly rated loans. In several cases the capital charge of a loan was significantly higher than its exposure, in accordance with the theoretical result mentioned above. This clearly demonstrates that the use of risk measurement and capital allocation methods, which violate basic theoretical requirements, can lead to an incorrect risk assessment and consequently to wrong business decisions.

*The views expressed in this paper are those of the authors and do not necessarily reflect the position of Deutsche Bank AG.

[1]We refer to Bluhm et al. (2002) and Crouhy et al. (2000) for a survey on credit portfolio modeling.

In Merton-Type-Models, there is a major obstacle to the application of expected shortfall allocation to credit portfolios. Since the loss distributions of the entire portfolio and single transactions are not tractable in analytical form one has to utilize Monte Carlo simulation to calculate expected shortfall contributions. It is easy to see that due to statistical fluctuations the simulation-based estimation of this conditional expectation is a demanding computational problem.[2]

In this article we introduce an importance sampling technique for Merton-Type-Models, which utilizes the infinite granularity approximation (compare to Vasicek (2002) and Gordy (2003)). Since we use this approximation technique only for importance sampling but not in the actual simulation of the portfolio model our method is applicable to portfolios of any size and structure. Applied to a test portfolio of 25000 loans it reduces the variance of the Monte Carlo estimate of expected shortfall - and therefore the number of required simulations - by a factor of 400. We present statistics which show that this technique calculates expected shortfall contributions for realistic portfolios with satisfactory numerical accuracy.

2 Risk measurement and capital allocation in credit portfolios

2.1 The credit portfolio model

For the sake of simplicity we perform our analysis in the framework of a one-period, default-only mode structural model with deterministic exposure-at-default. The results as well as the importance sampling technique can be easily generalized to models which incorporate rating migration.

The credit portfolio P consists of n loans. With each loan we associate an "Ability-to-Pay" variable $A_i : \mathbb{R}^{m+1} \to \mathbb{R}$, which is a linear combination of the m systematic variables $x_1, \dots, x_m$ and a specific variable z_i:

$$A_i(x_1, \dots, x_m, z_i) := \sum_{j=1}^{m} \phi_{ij} x_j + \sqrt{1 - R_i^2}\, z_i \tag{1}$$

with $0 \le R_i^2 \le 1$ and weight vector $(\phi_{i1}, \dots, \phi_{im})$. The loan loss $L_i : \mathbb{R}^{m+1} \to \mathbb{R}$ and the portfolio loss function $L : \mathbb{R}^{m+n} \to \mathbb{R}$ are defined by

$$L_i := l_i \cdot \mathbf{1}_{\{A_i \le D_i\}}, \quad L := \sum_{i=1}^{n} L_i,$$

where $0 < l_i$ and $D_i \in \mathbb{R}$ are the exposure-at-default and the default threshold respectively. As probability measure $\mathbb{P}$ on $\mathbb{R}^{m+n}$ we use the product measure

$$\mathbb{P} := N_{\mathbf{0},C} \times \prod_{i=1}^{n} N_{0,1},$$

where $N_{0,1}$ is the standardized one-dimensional normal distribution, $N_{\mathbf{0},C}$ the m-dimensional normal distribution with mean $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^m$ and covariance matrix $C \in \mathbb{R}_m^m$ and $n_{\mathbf{0},C}$ the density of $N_{\mathbf{0},C}$. We assume that the weight vector $(\phi_{i1}, \dots, \phi_{im})$ has been normalized in such a way that the variance of A_i is 1. Hence, the default probability p_i of the i-th counterparty equals

$$p_i := \mathbb{P}(A_i \le D_i) = N(D_i),$$

where N denotes the standardized one-dimensional normal distribution function. This relation is used to determine the default threshold from empirical default probabilities.

[2]We refer to Kurth and Tasche (2003) for a computational approach to expected shortfall in the analytic framework of CreditRisk+.

2.2 Risk Measures and Capital Allocation

After JP Morgan made its RiskMetrics system public in 1994 value-at-risk became the dominant concept for risk measurement. The value-at-risk $\mathrm{VaR}_\alpha(L)$ of L at level $\alpha \in (0,1)$ is defined as an α-quantile of L. More precisely, in this paper

$$\mathrm{VaR}_\alpha(L) := \inf\{x \in \mathbb{R} \mid \mathbb{P}(L \le x) \ge \alpha\}$$

is the smallest α-quantile. As long as the loss distributions of portfolios are normally distributed the VaR methodology encourages diversification, i.e.

$$\mathrm{VaR}(X+Y) \le \mathrm{VaR}(X) + \mathrm{VaR}(Y) \tag{2}$$

for normal loss distributions X and Y. However, for credit portfolios the normal distribution assumption is not justified and in this case subadditivity (2) no longer holds for value-at-risk. Diversification, which is commonly considered as a way to reduce risk, might increase value-at-risk.

Another disadvantage of value-at-risk is that the allocation of portfolio VaR to subportfolios and individual facilities is difficult. The standard solution in credit portfolio modeling is to allocate portfolio VaR proportional to the covariances

$$\mathrm{Cov}(L_1, L), \dots, \mathrm{Cov}(L_n, L). \tag{3}$$

This allocation technique, called volatility allocation, is the natural choice in classical portfolio theory where portfolio risk is measured by standard deviation (or volatility).

In general, combining volatility allocation with value-at-risk works well as long as all loss distributions are close to normal. However, for credit portfolios it does not: the capital allocated to a subportfolio $\bar{P}$ of P might be greater than the risk capital of $\bar{P}$ considered as a stand-alone portfolio, the capital charge of a loan might even be higher than its exposure. The reason for these nonsensical results is that value-at-risk and volatility reflect different properties of distributions. Value-at-risk is determined by a - usually rather extreme - point in the tail whereas volatility and covariances are sensitive to all parts of the distributions.

An alternative risk measure is expected shortfall (see, for instance, Rockafellar and Uryasev, 2000; Acerbi and Tasche, 2002). For practical purposes, the expected shortfall of L at level $\alpha \in (0,1)$, denoted by $\mathrm{ES}_\alpha(L)$, can be defined as

$$\mathrm{E}(L|L > \mathrm{VaR}_\alpha(L)) = (1-\alpha)^{-1}\mathrm{E}(L \cdot \mathbf{1}_{\{L>\mathrm{VaR}_\alpha(L)\}}). \tag{4}$$

Intuitively, expected shortfall can therefore be interpreted as the average of all losses above a given quantile of the loss distribution. Its exact definition[3] takes care of jumps of the loss distribution at its quantile.

An important advantage of expected shortfall is the simple allocation of risk capital to subportfolios or individual transactions: we will show that the expected shortfall contribution of the i-th loan should be defined as[4]

$$\mathrm{E}(L_i|L > \mathrm{VaR}_\alpha(L)) = (1-\alpha)^{-1}\mathrm{E}(L_i \cdot \mathbf{1}_{\{L>\mathrm{VaR}_\alpha(L)\}}). \tag{5}$$

Hence, the expected shortfall contribution of a loan can be considered as its average contribution to portfolio losses above quantile $\mathrm{VaR}_\alpha(L)$.

[3]More precisely: $\mathrm{ES}_\alpha(L) := (1-\alpha)^{-1}(\mathrm{E}(L\,\mathbf{1}_{\{L>\mathrm{VaR}_\alpha(L)\}}) + \mathrm{VaR}_\alpha(L)\cdot(\mathbb{P}(L \le \mathrm{VaR}_\alpha(L)) - \alpha))$. It is easy to see that for most loss distributions the expected shortfall ES_α is dominated by the first term given by (4). The second term is added to ensure coherence as defined in Artzner et al. (1997, 1999).

[4]Exactly: $\mathrm{ESC}_\alpha(L_i, L) := (\mathrm{E}(L_i \cdot \mathbf{1}_{\{L>\mathrm{VaR}_\alpha(L)\}}) + \beta_L \cdot \mathrm{E}(L_i \cdot \mathbf{1}_{\{L=\mathrm{VaR}_\alpha(L)\}}))/(1-\alpha)$ with $\beta_L := \frac{\mathbb{P}(L \le \mathrm{VaR}_\alpha(L)) - \alpha}{\mathbb{P}(L=\mathrm{VaR}_\alpha(L))}$ which again is dominated by (5).

In contrast to volatility contributions, the allocation of expected shortfall is linear and diversifying. As a consequence the capital charge of a loan cannot exceed its exposure. We will now substantiate these claims within an axiomatic framework for the analysis and comparison of allocation schemes. In particular, we will demonstrate that the existence of linear, diversifying capital allocations is mathematically equivalent to certain properties of the associated risk measure. As a consequence there is no linear diversifying allocation for VaR but for expected shortfall.[5]

2.3 Axiomatic foundation of capital allocation

We introduce V as a linear subspace of the space of real-valued random variables on the probability space Ω representing the possible states of the world. By identifying a portfolio X with its loss function we can interpret V as the space of portfolios. A capital allocation is a function Λ from $V \times V$ to $\mathbb{R}$. Its interpretation is that $\Lambda(X, Y)$ represents the capital allocated to the portfolio X, considered as a subportfolio of portfolio Y. We propose the following requirements for a capital allocation scheme in a financial institution.

- The capital allocated to a union of subportfolios is equal to the sum of the capital amounts allocated to the individual subportfolios. In particular, the risk capital of a portfolio equals the sum of the risk capital of its subportfolios.
- The capital allocated to a subportfolio X of a larger portfolio Y never exceeds the risk capital of X considered as a stand-alone portfolio.
- A small increase in a position does only have a small effect on the risk capital allocated to that position.

The following definition formalizes these principles.

Definition 1 The capital allocation Λ is called

$$\begin{array}{lll} \textit{linear :} & \Lambda(aX + bY, Z) = a\Lambda(X, Z) + b\Lambda(Y, Z) & \forall a, b \in \mathbb{R}, X, Y, Z \in V, \\ \textit{diversifying:} & \Lambda(X, Y) \leq \Lambda(X, X) & \forall X, Y \in V, \\ \textit{continuous at } Y\textit{:} & \lim_{\epsilon \to 0} \Lambda(X, Y + \epsilon X) = \Lambda(X, Y) & \forall X \in V. \end{array}$$

The real valued function ρ on V, defined by

$$\rho(X) = \Lambda(X, X),$$

is called the risk measure associated with Λ. It turns out that a linear and diversifying capital allocation, which is continuous at a portfolio $Y \in V$, is uniquely determined by its associated risk measure, i.e. the diagonal values of Λ. More specifically, given the portfolio Y then the capital allocated to a subportfolio X of Y is the derivative of the associated risk measure ρ at Y in the direction of X.

Theorem 1 *Let Λ be a linear, diversifying capital allocation. If Λ is continuous at $Y \in V$ then for all $X \in V$*

$$\Lambda(X, Y) = \lim_{\epsilon \to 0} \frac{\rho(Y + \epsilon X) - \rho(Y)}{\epsilon}.$$

This theorem shows that the three axioms are sufficient to uniquely determine a capital allocation scheme. It remains to be shown that capital allocations, which satisfy the axioms, do exist. It turns out that the existence of linear, diversifying capital allocations is mathematically equivalent to certain properties of the associated risk measure. We consider risk measures as real-valued functions on V and introduce the following attributes.

[5] We refer to Kalkbrener (2002) for proofs of the results presented in the following subsection. Additional information on capital allocation for coherent risk measures can be found in Delbaen (2000), Denault (2001), Overbeck (1999), Bluhm et al. (2002).

Definition 2 The risk measure $\rho : V \to \mathbb{R}$ is called

$$\begin{array}{llll} \textit{positively homogeneous:} & \rho(aX) & = & a\rho(X) \quad \forall a \geq 0,\ X \in V, \\ \textit{sub-additive:} & \rho(X+Y) & \leq & \rho(X)+\rho(Y) \quad \forall X, Y \in V. \end{array}$$

Now let V^* be the set of real linear functions on V and for a given risk measure ρ consider the following subset

$$H_\rho := \{h \in V^* \mid h(X) \leq \rho(X) \text{ for all } X \in V\}.$$

Provided that ρ is positively homogeneous and sub-additive then for every $Y \in V$ there exists an $h_Y^\rho \in H_\rho$ with $h_Y^\rho(Y) = \rho(Y)$. This allows to define a capital allocation Λ_ρ by

$$\Lambda_\rho(X, Y) := h_Y^\rho(X). \tag{6}$$

The set H_ρ can be interpreted as a collection of (generalized) scenarios. In this picture $h(Y)$ is the loss of portfolio Y under scenario h. Note that scenarios are linear functions on portfolios: the loss of a union of portfolios under a scenario is equal to the sum of the losses of the individual portfolios. The key result stated above is the fact that the application of the risk measure ρ to portfolio Y can be expressed by the generalized scenario h_Y^ρ. The capital allocated to a subportfolio X of portfolio Y is then simply the loss of X under scenario h_Y^ρ.

The following theorem states the equivalence between positively homogeneous, sub-additive risk measures and linear, diversifying capital allocations.

Theorem 2 **(a)** *If there exists a linear, diversifying capital allocation Λ with associated risk measure ρ then ρ is positively homogeneous and sub-additive.*
(b) *If ρ is positively homogeneous and sub-additive then Λ_ρ is a linear, diversifying capital allocation with associated risk measure ρ.*

This theorem implies that in the general case, in particular for credit portfolios, there do not exist linear diversifying capital allocations for VaR since VaR is not sub-additive. Expected shortfall ES, on the other hand, is a coherent risk measure and therefore positively homogeneous and sub-additive. Hence, application of (6) to expected shortfall yields a linear, diversifying capital allocation with associated risk measure ES. The scenario function $h_Y^{\mathrm{ES}}(X)$ for this risk measure is given by $\mathrm{E}(X \cdot g(Y))$ with[6]

$$g(Y) := (1-\alpha)^{-1}(\mathbf{1}_{\{Y > \mathrm{VaR}_\alpha(Y)\}} + \beta_Y \mathbf{1}_{\{Y = \mathrm{VaR}_\alpha(Y)\}}), \tag{7}$$

where β_Y is a real number and

$$\beta_Y := \frac{\mathbb{P}(Y \leq \mathrm{VaR}_\alpha(Y)) - \alpha}{\mathbb{P}(Y = \mathrm{VaR}_\alpha(Y))} \quad \text{if} \quad \mathbb{P}(Y = \mathrm{VaR}_\alpha(Y)) > 0.$$

The exact form of expected shortfall contributions (5) immediately follows from the general construction principle (6) and formula (7).

3 Numerical Results

In the last section we have compared expected shortfall contributions and volatility contributions from an axiomatic point of view. In this section we will analyze the practical consequences by applying both allocation schemes to a realistic credit portfolio.

We use a test portfolio consisting of 25000 loans with an inhomogeneous exposure and default probability distribution. The average exposure size is 0.004% of the total exposure

[6] Note that $\mathrm{ES}_\alpha(Y) = \mathrm{E}(Y \cdot g(Y))$ and $\mathrm{ES}_\alpha(X) \geq \mathrm{E}(X \cdot g(Y))$ for every $X, Y \in V$.

and the standard deviation of the exposure size is 0.026%. Default probabilities vary between 0.02% and 27%. The portfolio expected loss is 0.72% and the unexpected loss, i.e. the standard deviation, is 0.87%. Default correlations are specified by the KMV factor model (see www.kmv.com or Baestaens and van den Bergh (1997) and Overbeck and Stahl (2003) for a description of the model), comprising 96 systematic country and industry factors.

For this test portfolio we have calculated the risk measures $\mathrm{VaR}_{0.9998}(L)$, $\mathrm{ES}_{0.999}(L)$ and $\mathrm{ES}_{0.99}(L)$. The $\mathrm{VaR}_{0.9998}(L)$ is the typical risk measure used to determine the capital requirement in a financial institution, which aims at a AAA-rating. The $\mathrm{ES}_{0.999}(L)$ has been chosen since it leads to a comparable figure, while being based on a coherent risk measure. The $\mathrm{ES}_{0.99}(L)$ was calculated to study the impact of the location of the threshold on the properties of the expected shortfall measure. Importance sampling improved (see next section) Monte-Carlo simulation leads to the following results

$$\begin{array}{lcl} \mathrm{VaR}_{0.9998}(L) & = & 10.50\% \\ \mathrm{ES}_{0.999}(L) & = & 9.43\% \\ \mathrm{ES}_{0.99}(L) & = & 5.68\% \end{array}$$

In the next step the portfolio capital is distributed to the individual loans using the different capital allocation algorithms. The $\mathrm{VaR}_{0.9998}(L)$ is distributed using volatility contributions (3), the $\mathrm{ES}_{0.999}(L)$ and $\mathrm{ES}_{0.99}(L)$ are distributed using expected shortfall contributions (5). In all our figures we look at the relative capital charge, i.e. the ratio of the allocated capital and the exposure-at-default.

3.1 Default risk of individual loans

Figure 1 displays the 50 loans with the highest capital charge under expected shortfall allocation based on the 99.9% quantile. For all loans the relation

$$\mathrm{VaR}_{0.9998}(L) > \mathrm{ES}_{0.999}(L) > \mathrm{ES}_{0.99}(L)$$

holds, which is also the order of the three different capital figures distributed. But we notice that the order of the capital consumption changes. For instance, the loan contributing the most to the 99.9% expected shortfall is only the fourth largest if capital is allocated proportional to covariances. Also the absolute difference in capital is high. The highest capital consumption for expected shortfall is 93% of the exposure compared to almost 200% for covariances. In the context of RAROC-pricing it is conceivable that some of these loans earn enough margin to cover the capital costs based on expected shortfall but not on volatility contributions.

More strikingly, we find that under the covariance allocation the capital charge exceeds 100% for almost all loans in this sub-sample. This means that the capital charge of these loans is higher than their exposure, demonstrating that the inconsistencies of the covariance allocation are not purely theoretical, but apply to realistic credit portfolios.

In figure 2 we perform the same comparison for a different sub-sample of loans, here we look at a subset of loans with a AA+, i.e. the highest rating in our test portfolio. In contrast to the previous results, expected shortfall contributions based on the 99.9% quantile are higher than volatility contributions for these loans, even though the total expected shortfall capital is smaller. Put into a RAROC context this means that under expected shortfall allocation loans to AA+ obligors need to earn a higher margin to exceed a risk-adjusted return threshold than under volatility allocation. The results in figures 1 and 2 illustrate that unrealistically high capital charges for poorly rated loans are avoided under expected shortfall allocation by distributing a higher proportion of the portfolio capital to highly rated loans. As a consequence the expected shortfall method produces a characteristically different relative distribution of capital when compared to the covariance method. This can

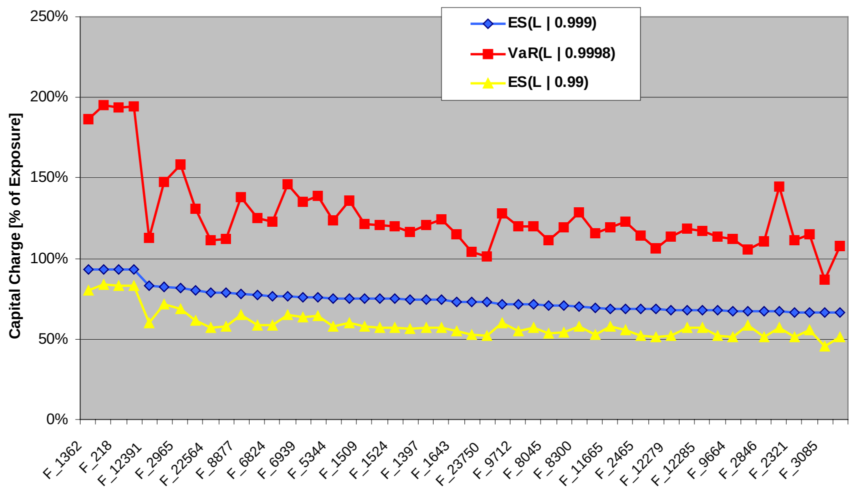

Figure 1: Comparison between expected shortfall and covariance capital allocation for loans with highest capital charges.

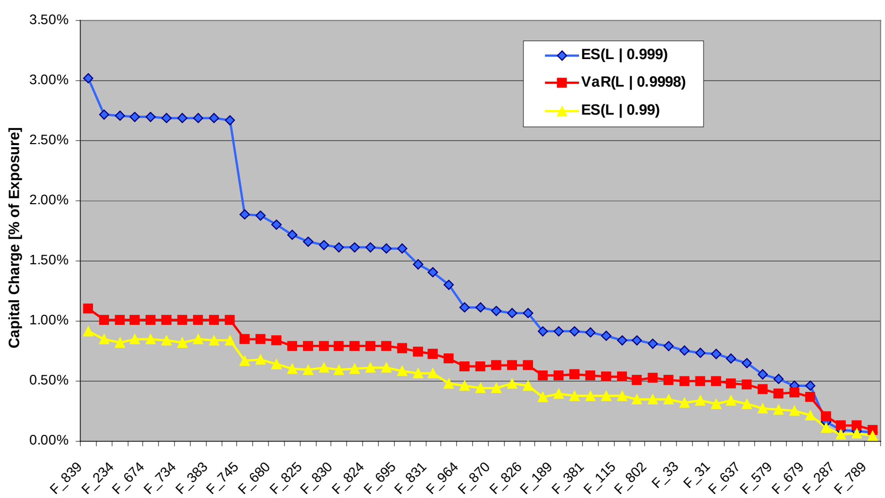

Figure 2: Comparison between expected shortfall and covariance capital allocation for AA+ loans.

be seen in figure 3 where we plot the normalized average capital charge per rating class. Normalized here means that the distributed capital for each method has been normalized to a common value. This chart clearly demonstrates that all three capital allocation methods scale with credit quality. However, if expected shortfall is based on a high quantile it is less sensitive to credit quality than volatility allocation.[7]

3.2 Concentration risk

We will now investigate the impact of the second main risk driver in portfolios, namely concentration risk. This risk is caused by default correlations and name concentration.

Expected shortfall contributions measure the average contribution of individual loans to portfolio losses above a specified α-quantile. For a high α these losses are mainly driven by default correlations and name concentration and expected shortfall allocation therefore is - almost by definition - very sensitive to concentration risk.

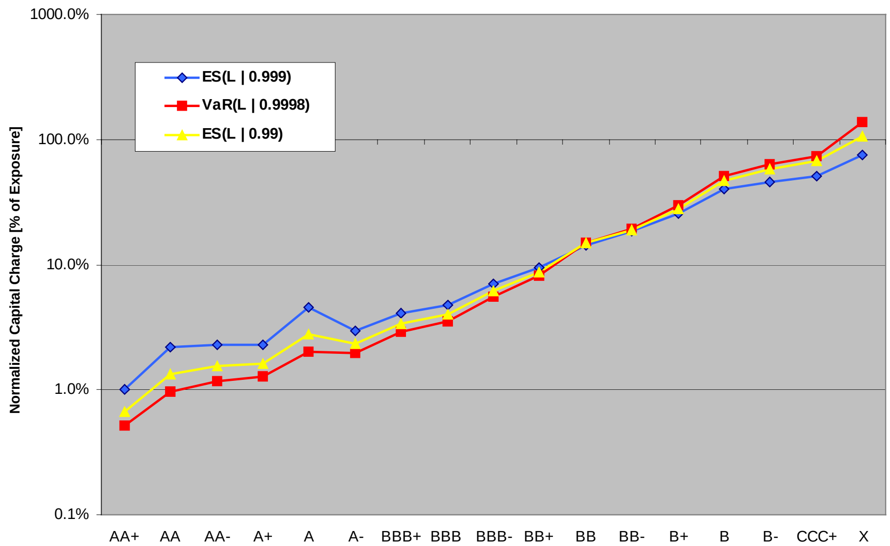

Figure 3: Normalized capital charge for different rating classes.

To support this statement we have isolated a set of loans with identical default probabilities (AA+) and factor weights, but different R^2 parameter.[8] Figure 4 again displays the normalized capital charge as percentage of exposure. This plot shows that capital allocation based on expected shortfall penalizes concentration risks more strongly than the covariance method. For instance, the 99.9% expected shortfall contribution at $R^2 = 60\%$ is three times higher than at $R^2 = 30\%$ whereas the volatility contribution not even doubles, i.e. from 1.4% to 4.2% (shortfall) versus from 0.7% to 1.25% (volatility) in the R^2-range $30\% - 60\%$. Again this effect clearly depends on the choice of the quantile for expected shortfall. The higher the quantile the steeper the increase of the capital charge with increasing correlation. In the limit of the 0% quantile the curve is flat since this limit corresponds to expected loss distribution, which is independent of correlation.

The test portfolio shows a high exposure concentration in rating class A. Note that the correlation effect also explains the local peak in this rating class in figure 3.

[7]Note that the curves in figure 3 are displayed on a logarithmic scale which makes differences look smaller than they really are (compare to figures 1 and 2).

[8]Note that the R^2-parameter is the coupling of the loan to the systematic factors and therefore quantifies the correlation of the loan with the rest of the portfolio.

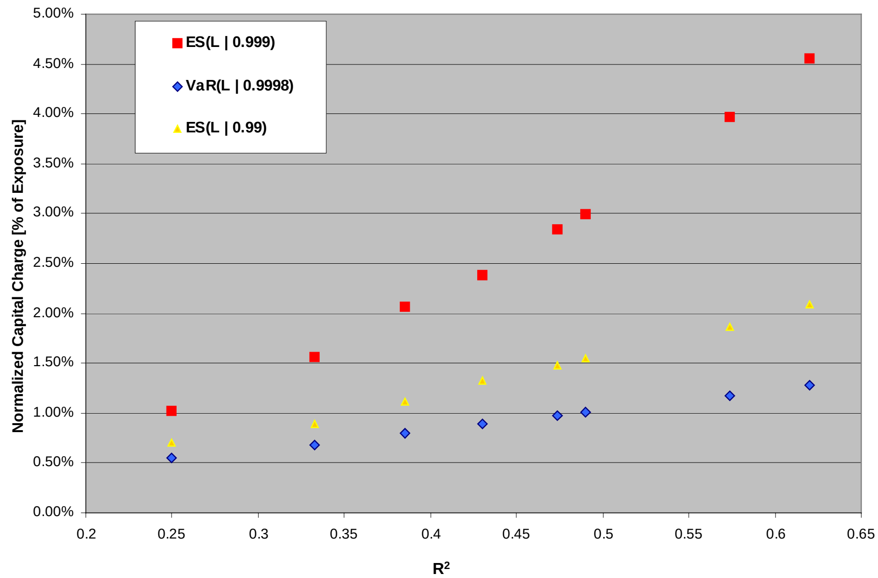

Figure 4: Normalized capital charge as a function of correlation.

4 Importance sampling

We use Monte Carlo simulation for the actual calculation of risk capital at portfolio and transaction level. The main practical problem in applying expected shortfall to realistic credit portfolios is the computation of numerically stable MC estimates. In this section we adapt importance sampling to our credit portfolio model and show that this technique significantly reduces the variance of Monte Carlo simulation.

4.1 Straightforward Monte Carlo simulation

The efficient computation of expected shortfall (4) and expected shortfall contributions (5) is a challenging task for realistic portfolios. Straightforward Monte Carlo simulation does not work well (see figure 6). As an example, assume that we want to compute expected shortfall with respect to the $\alpha = 99.9\%$ quantile and compute $\nu = 100000$ MC samples $s_1 \geq s_2 \geq \ldots \geq s_\nu$ of the portfolio loss L. Then $\mathrm{ES}_\alpha(L)$ becomes

$$(1-\alpha)^{-1}\mathrm{E}(L \cdot \mathbf{1}_{\{L>c\}}) = (1-\alpha)^{-1} \int L \cdot \mathbf{1}_{\{L>c\}}\, d\mathbb{P} = \sum_{i=1}^{100} s_i/100, \tag{8}$$

where $c := \mathrm{VaR}_\alpha(L)$. Since the computation of $\mathrm{ES}_\alpha(L)$ is only based on 100 samples it is subject to large statistical fluctuations numerically unstable. This is even more true for expected shortfall contributions of individual loans. A significantly higher number of samples has to be computed which makes straightforward MC simulation impracticable for large credit portfolios.

4.2 Monte Carlo simulation based on importance sampling

Importance sampling is a technique for reducing the variance of MC simulations and - as a consequence - the number of samples required for stable results. It has been successfully applied to problems in market risk (Glasserman et al., 1999). In our setting, the integral in (8) is replaced by the equivalent integral

$$\int L \cdot \mathbf{1}_{\{L>c\}} \cdot f \, d\bar{\mathbb{P}}, \tag{9}$$

where $\mathbb{P}$ is continuous with respect to the probability measure $\bar{\mathbb{P}}$ and has density f. The objective is to choose $\bar{\mathbb{P}}$ in such a way that the variance of the Monte-Carlo estimate for the integral (9) is minimal under $\bar{\mathbb{P}}$. This MC estimate is

$$\mathrm{ES}_\alpha(L)_{\nu,\bar{\mathbb{P}}} := \frac{1}{\nu}\sum_{i=1}^{\nu} L_{\bar{\mathbb{P}}}(i) \cdot \mathbf{1}_{\{L_{\bar{\mathbb{P}}}(i)>c\}} \cdot f(i), \tag{10}$$

where $L_{\bar{\mathbb{P}}}(i)$ is a realization of the portfolio loss L under the probability measure $\bar{\mathbb{P}}$ and $f(i)$ is the corresponding value of the density function. Under suitable conditions as $\nu \to \infty$, $\mathrm{ES}_\alpha(L)_{\nu,\bar{\mathbb{P}}}$ converges to (9) and the sampling error converges as

$$\sqrt{\nu} \cdot (\mathrm{ES}_\alpha(L)_{\nu,\bar{\mathbb{P}}} - \mathrm{ES}_\alpha(L)) \xrightarrow{d} N(0, \sigma_{\mathrm{ES}_\alpha(L)}(\bar{\mathbb{P}})), \tag{11}$$

where $\sigma^2_{\mathrm{ES}_\alpha(L)}(\bar{\mathbb{P}})$ is the variance of $L \cdot \mathbf{1}_{\{L>c\}} \cdot f$ under $\bar{\mathbb{P}}$, i.e.

$$\sigma^2_{\mathrm{ES}_\alpha(L)}(\bar{\mathbb{P}}) = \int \left(L \cdot \mathbf{1}_{\{L>c\}} \cdot f\right)^2 d\bar{\mathbb{P}} - \left(\int L \cdot \mathbf{1}_{\{L>c\}} \cdot d\mathbb{P}\right)^2. \tag{12}$$

In the following we restrict the set of probability measures $\bar{\mathbb{P}}$, which we consider to determine a minimum of (12): for every $M = (M_1, \dots, M_m) \in \mathbb{R}^m$ define the probability measure $\mathbb{P}_M$ by[9]

$$\mathbb{P}_M := N_{M,C} \times \prod_{i=1}^{n} N_{0,1}, \tag{13}$$

where $N_{M,C}$ is the m-dimensional normal distribution with mean M and covariance matrix C. In other words, those probability measures are considered which only change the mean of the systematic components $x_1, \dots, x_m$ in the definition of the "Ability-to-Pay" variables $A_1, \dots, A_n$. This choice is motivated by the nature of the problem. The MC estimate of integral (9) can be improved by increasing the number of scenarios which lead to high portfolio losses, i.e. portfolio losses above threshold c. This can be realized by generating a sufficiently large number of defaults in each sample. Since defaults occur when "Ability-to-Pay" variables fall below default thresholds we can enforce a high number of defaults by adding a negative mean to the systematic components.

4.3 Shifting the systematic mean

Having thus restricted importance sampling to measures of the form (13) we can formulate the minimization condition as

$$\partial_{M_i} \sigma^2_{\mathrm{ES}_\alpha(L)}(\mathbb{P}_M) = 0, \;\; \forall\, i = 1, \dots, m. \tag{14}$$

Using the representation in (9) and the specification of the portfolio model this leads to the system of m equations

$$2\sum_{j=1}^{m} C^{-1}_{ij} M_j = -\partial_{M_i} \log\left(\int L(x-M,z)^2 \cdot \mathbf{1}_{\{L(x-M,z)>c\}} \, dN_{\mathbf{0},C}(x) \prod_{i=1}^{n} dN_{0,1}(z_i)\right).$$

[9] Note that the initial measure $\mathbb{P}$ equals $\mathbb{P}_0$.

and the explicit representation of the portfolio loss reads

$$L(x,z) = \sum_{i=1}^{n} l_j \cdot \mathbf{1}_{\{N^{-1}(p_i) > \sum_{k=1}^{m} \phi_{ik} x_k + \sqrt{1-R_i^2} z_i\}}. \tag{15}$$

For realistic portfolios with thousands of loans this system is analytically and numerically intractable.

To progress we therefore approximate the original portfolio P by a homogeneous and infinitely granular portfolio $\bar{P}$. This means that the losses, default probabilities and "Ability-to-Pay" variables of all loans in $\bar{P}$ are identical and that the number of loans n is infinite with fixed total exposure.[10] There is no unique procedure to establish the homogeneous portfolio, which is closest to a given portfolio. A well know technique consists of matching the second moment of the loss distribution as employed in the construction of comparable portfolios in Gordy (2003). In our model, however, the calculation of the second moment becomes very time-consuming for large portfolios. In the appendix we propose a fast technique for determining the parameters of the homogeneous portfolio $\bar{P}$, i.e. exposure-at-default l, default probability p, R^2 and the set of factor weights ρ_j, $(j = 1, \ldots, m)$.

The loss function $L_m^\infty : \mathbb{R}^m \to \mathbb{R}$ of the infinite homogeneous portfolio $\bar{P}$ has the form

$$L_m^\infty(x_1, \ldots, x_m) := l \cdot N\left(\frac{N^{-1}(p) - \sum_{j=1}^{m} \rho_j x_j}{\sqrt{1-R^2}}\right)$$

and the threshold c^∞ is defined by $c^\infty := \mathrm{VaR}_\alpha(L_m^\infty)$. The approximation of P by $\bar{P}$ has transformed the variance reduction problem for the finite inhomogeneous portfolio to the variance reduction problem for an infinite homogeneous portfolio, i.e. the problem of computing a minimum $M = (M_1, \ldots, M_m) \in \mathbb{R}^m$ of the variance $\sigma^2_{\mathrm{ES}_\alpha(L_m^\infty)}(M)$ of

$$L_m^\infty \cdot \mathbf{1}_{\{L_m^\infty > c^\infty\}} \cdot \frac{n_{\mathbf{0},C}}{n_{M,C}}$$

with respect to $N_{M,C}$. Note that we have achieved a significant reduction of complexity: the dimension of the underlying probability space has been reduced from $m+n$ to m and the loss function L_m^∞ is not a large sum but has a concise analytic form. We will now present an efficient procedure which computes the minimum of $\sigma^2_{\mathrm{ES}_\alpha(L_m^\infty)}(M)$ with arbitrary precision (see Kalkbrener et al. (2003) for a correctness proof of the algorithm). The computation is done in two steps:

One-factor model. Instead of m systematic factors $x_1, \ldots, x_m$ we consider the corresponding one-factor model and compute the minimum $M^{(1)} \in \mathbb{R}$ of the variance $\sigma^2_{\mathrm{ES}_\alpha(L_1^\infty)}(M)$ in the case $m = 1$: it is the minimum of

$$\int_{-\infty}^{N^{-1}(1-\alpha)} \frac{(L_1^\infty \cdot n_{0,1})^2}{n_{M,1}}\, dx, \qquad \text{where } L_1^\infty(x) := l \cdot N\left(\frac{N^{-1}(p) - \sqrt{R^2}x}{\sqrt{1-R^2}}\right).$$

Multi-factor model: The one-dimensional minimum $M^{(1)}$ can be lifted to the m-dimensional minimum $M^{(m)} = (M_1^{(m)}, \ldots, M_m^{(m)})$ of $\sigma^2_{\mathrm{ES}_\alpha(L_m^\infty)}(M)$ by

$$M_i^{(m)} := \frac{M^{(1)} \cdot \sum_{j=1}^{m} \mathrm{Cov}(x_i, x_j) \cdot \rho_j}{\sqrt{R^2}} \quad \forall\, i = 1, \ldots, m. \tag{16}$$

[10] We emphasize that this approximation technique is only used for determining a mean vector M for importance sampling. The actual calculations of expected shortfall and expected shortfall contributions are based on Monte Carlo simulation of the full portfolio model as specified in section 2.1.

4.4 Analysis

Using the homogeneous portfolio approximation we have been able to derive the shift vector $(M_1^{(m)}, \ldots, M_m^{(m)})$ in (16), which reduces the Monte-Carlo sampling fluctuation, for our proposed importance sampling density. We do not know yet, however, whether this result leads to a significant error reduction in the Monte-Carlo based estimation of expected shortfall for a realistic portfolio. We therefore applied importance sampling to our sample portfolio of 25000 loans (see section "Numerical Results") and calculated the standard deviation of the Monte-Carlo estimator for the portfolio risk measure expected shortfall $\mathrm{ES}_{0.999}(L)$. In figure 5 we plot the standard deviation of the Monte-Carlo estimator as a function of the norm of the vector $(M_1^{(m)}, \ldots, M_m^{(m)})$. A scaling factor of 0 corresponds to no importance sampling, whereas a scaling factor of 1 corresponds to $(M_1^{(m)}, \ldots, M_m^{(m)})$. The other points represent vectors with identical direction but a linearly interpolated/extrapolated norm, i.e. 0.5 corresponds to the vector $0.5 \cdot (M_1^{(m)}, \ldots, M_m^{(m)})$. These results were obtained from a sample of 40 independent Monte-Carlo runs of 10000 simulation trials for each scaling factor. On the right-hand axis we plot the average result from the 40 runs with the standard deviation in error bars.

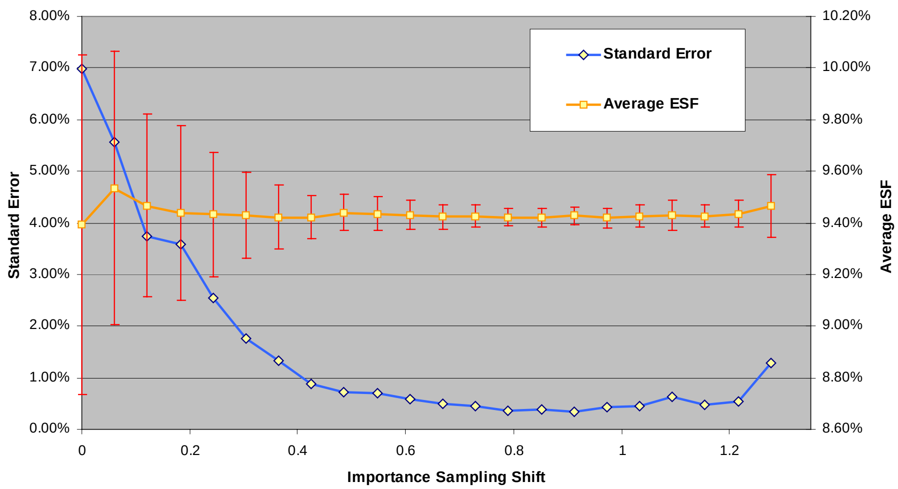

Figure 5: Monte-Carlo sampling error as a function of the importance sampling shift.

From these results we draw two conclusions. First we have demonstrated that importance sampling can significantly improve the quality of the Monte-Carlo estimate of the expected shortfall measure. The variance ratio between the optimal point in the graph and the no-shift case is 400, i.e. the same precision without any importance sampling would require 400 times more simulations. Improvements of a comparable magnitude were found for the Monte-Carlo estimate of the quantile of the loss distribution, i.e. the value-at-risk.

Secondly, we observe that our theoretical optimal shift size slightly overestimates the empirical optimal shift. Our explanation for this is that in our determination of the equivalent homogeneous portfolio we have overestimated the average correlation R^2. This has been confirmed by the observation that the R^2 determined from fitting a Vasicek distribution to our best Monte Carlo estimate for the 2bp quantile is approximately 10% smaller than the one calculated from the approximation procedure outlined in the appendix.

In figure 6 we demonstrate the effect of introducing the importance sampling density on the quality of expected shortfall contributions of individual loans. For all 25000 loans in our sample portfolio we have estimated the statistical error of the expected shortfall contributions from 10 independent runs of 400000 simulation trials, with and without the importance sampling mean. On the horizontal axis in figure 6 we divide this statistical error in bins and on the vertical axis we display the number of loans with a standard error in the corresponding bin. Note that the bins on the horizontal axis are not of equal size. These

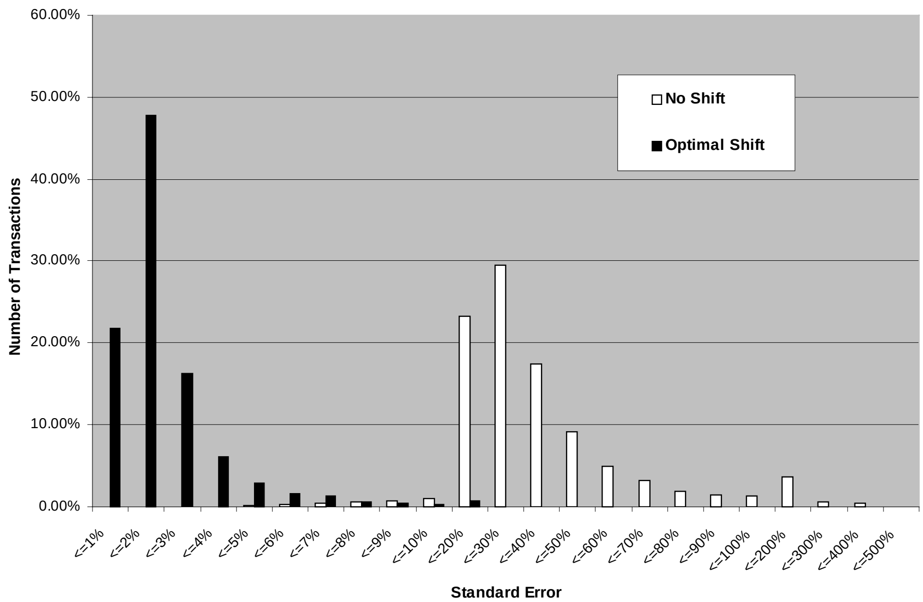

Figure 6: Distribution of statistical fluctuation of the $ES_{0.999}$ estimate with 400000 simulation trials.

results show that using importance sampling we can allocate expected shortfall at transaction level with satisfactory accuracy. Without importance sampling, on the other hand, the statistical fluctuations are so high that expected shortfall contributions of individual loans are inaccessible.

5 Conclusion

To summarize, we have investigated the properties of different capital allocation schemes both from a theoretical and a practical perspective. The theoretical analysis starts with a mathematical formulation of the desired properties of a sensible capital allocation. In this framework it is possible to rigorously prove that the expected shortfall risk measure leads to a diversifying capital allocation whereas the classical portfolio theory inspired volatility allocation is not consistent with diversification. In practice this deficiency is exhibited in capital charges in excess of 100%, i.e. the loan as part of the portfolio utilizes more capital than it would on a stand-alone basis. Monte-Carlo simulation of a large, realistic loan portfolio shows that besides being theoretically appealing expected shortfall also performs well in a practical application: it correctly scales with the credit riskiness of the loans and it exhibits strong sensitivity on correlations, i.e. it is well suited to detect concentration risks. This property is due to the fact that the expected shortfall quantifies the contribution of a loan to a loss event in the region of the quantile. This region of the loss distribution is dominated by events in which many large loans default simultaneously, i.e. the default correlation is a crucial parameter here.

Our analysis of expected shortfall capital allocation was facilitated by a break-through in the Monte-Carlo simulation of loan portfolios. Guided by the approximate analytical solution to our credit portfolio model we defined an importance sampling algorithm, which reduces the variance of the Monte-Carlo estimates for portfolio expected shortfall by a factor of 400 and for expected shortfall contribution on average by a factor of 350. This technique does not only enable the calculation of expected shortfall allocation in practice, but it can also be used to substantially reduce the computing time necessary to calculate high quantiles of the credit loss distribution.

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Appendix: Homogeneous Portfolio Approximation

In a homogeneous portfolio all loans are specified by identical default probabilities, exposures-at-default and "Ability-to-Pay" variables. The following procedure approximates the original portfolio P by a homogeneous portfolio $\bar{P}$. The parameters of $\bar{P}$ are derived from the parameters in the original portfolio.

Loss and default probability. The homogeneous loss l is the average of the individual losses l_i and the homogeneous default probability p is the exposure-at-default weighted default probability of all loans in the portfolio:

$$l := \frac{\sum_{i=1}^n l_i}{n}, \qquad p := \frac{\sum_{i=1}^n p_i l_i}{\sum_{i=1}^n l_i}.$$

Weight vector. The homogeneous weight vector is the normalized, weighted sum of the weight vectors of the individual loans: in this paper the positive weights $g_1, \dots, g_n \in \mathbb{R}$ are given by $g_i := p_i l_i$, i.e. the i-th weight equals the i-th expected loss, and the homogeneous weight vector $\rho = (\rho_1, \dots, \rho_m)$ is defined by

$$\rho := \psi / s \quad \text{with} \quad \psi = (\psi_1, \dots, \psi_m) := \sum_{i=1}^n g_i \cdot (\phi_{i1}, \dots, \phi_{im}).$$

The scaling factor $s \in \mathbb{R}$ is chosen such that

$$R^2 = \sum_{i,j=1}^m \rho_i \cdot \rho_j \cdot \mathrm{Cov}(x_i, x_j) \tag{17}$$

holds, where R^2 is defined in (18).

$\boldsymbol{R^2}$. The specification of the homogeneous R^2 is based on the condition that the weighted sum of "Ability-to-Pay" covariances is identical in the original and the homogeneous portfolio. More precisely, define

$$R^2 := \frac{\sum_{k,l=1}^m \psi_k \psi_l \mathrm{Cov}(x_k, x_l) - \sum_{i=1}^n g_i^2 R_i^2}{(\sum_{i=1}^n g_i)^2 - \sum_{i=1}^n g_i^2} \tag{18}$$

and the i-th homogeneous "Ability-to-Pay" variable by

$$\bar{A}_i := \sum_{j=1}^m \rho_j x_j + \sqrt{1 - R^2} z_i.$$

The following equality holds for the weighted sum of "Ability-to-Pay" covariances of the original and the homogeneous portfolio:

$$\sum_{i,j=1}^n g_i g_j \mathrm{Cov}(A_i, A_j) = \sum_{i,j=1}^n g_i g_j \mathrm{Cov}(\bar{A}_i, \bar{A}_j).$$