Stress testing of credit portfolios in light- and heavy-tailed models

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Abstract

As, in light of the recent financial crises, stress tests have become an integral part of risk management and banking supervision, the analysis and understanding of risk model behaviour under stress has become ever more important. In this paper, we analyse asset correlations, default probabilities and default correlations under stress in a generalised Merton-type credit portfolio setup covering light- and heavy-tailed distributions. It turns out that the model behaviour under stress strongly depends on the heaviness of the tails of the risk factors. Contrary to popular belief light-tailed models show a higher impact in extreme stress scenarios. We use our results to study the implications for credit reserves and capital requirements under stress.

Keywords: stress testing, credit portfolio modelling, tail dependence

1 Introduction

In the aftermath of the subprime crisis and the European sovereign debt crisis, stress testing of bank portfolios has become an integral part of financial risk management and banking supervision. Stress tests for credit portfolios are of particular importance, since, in a typical bank, risk capital for credit risk far outweighs capital requirements for any other risk class. In this paper, we study the behaviour of light-tailed and heavy-tailed credit portfolio models under stress and investigate the implications for credit reserves and capital requirements.

Our analysis is performed in structural credit portfolio models. In this model class, the default event of each obligor is represented through an asset return (or ability-to-pay) variable and a default threshold. More precisely, an obligor defaults if its ability-to-pay variable falls below its default threshold. To reflect risk concentrations, each ability-to-pay variable is decomposed into a sum of systematic factors, which are often identified with geographic regions or industries, and an obligor-specific factor.

Although widely questioned, the industry standard is still to employ multivariate normally distributed risk factors and ability-to-pay variables. The normal distribution has been widely criticised, because (joint) extreme events are less probable under a normal distribution than observed in historical financial time series. This *tail behaviour* of a normal distribution is called *light-tailed*, as opposed to *heavy-tailed* distributions, which generate more extreme events and are generally considered more appropriate for modelling financial variables.[5;12]

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In order to cover a wide range of light-tailed to heavy-tailed distributions we consider a general family of distributions, so-called *normal variance mixtures*, which contains the normal distribution and the Student-t distribution as special cases. Loosely speaking, normal variance mixtures arise as normal random variables with a stochastic variance, that is, by multiplying the normally distributed factors with a strictly positive random variable W that specifies the stochastic variance. An attractive feature of normal variance mixtures is that the key parameters for describing dependence are the correlation matrix and the *tail index* of W, which specifies the tail dependence of the model.

The objective of this paper is to show that light-tailed and heavy-tailed models behave differently under stress. In a stress test, credit portfolios are typically evaluated under the assumption of adverse economic conditions. A natural way for implementing stress tests in structural credit portfolio models is to translate the stress scenario into constraints on risk factors. In our setup, the constraints are formalised by truncating risk factor variables, that is, by conditioning on the range of values that the risk factors may attain.

We study the impact of stress on asset correlations, default probabilities and default correlations. Aside from analysing the behaviour under stress for given stress levels or stress probabilities, we investigate the asymptotic behaviour, that is, the behaviour under stress as the stress level becomes arbitrarily high. Our results show that the impact of stress on the asymptotic behaviour is greater in light-tailed models than in heavy-tailed models, as formally proved in Kalkbrener and Packham [10] and Packham *et al.* [14]. More specifically,

- asset correlations under stress are less sensitive for heavy-tailed models than light-tailed models;
- default correlations under stress converge to 0 for light-tailed models and to a number strictly greater than 0 for heavy-tailed models;
- default probabilities converge to 1 for light-tailed models and to a number strictly smaller than 1 for heavy-tailed models.

These results have a number of interesting implications.

According to the concept of *tail dependence*, explained in Section 4, light-tailed models are asymptotically independent and are therefore perceived to be incapable of generating joint extreme events. However, our results show that, under extreme stress, default is a sure event for light-tailed models. We study and explain the relation between tail dependence and default under stress in Section 4.

We further study the usual risk measures – value-at-risk, expected loss and economic capital – under stress. It turns out that value-at-risk is smaller in light-tailed models than in heavy-tailed models for moderate stress, but greater in light-tailed models for extreme stress. The expected loss behaves in a similar way. Therefore, under extreme stress, light-tailed models react in a stronger way than heavy-tailed models. It should be noted, though, that extreme stress refers to stress probabilities that are much smaller than the probability of stress scenarios typically considered in risk management.

The paper is structured as follows: in Section 2, we give some background on credit portfolio risk and define stress tests in structural credit portfolio models. Section 3 considers asset correlations under stress. Default probabilities and default correlations under stress are treated in Section 4. Implications for credit reserves and capital requirements are analysed in Section 5.

2 Credit portfolio modelling

2.1 Structural credit portfolio models

Depending on their formulation, credit portfolio models can be divided into reduced-form models and structural (or firm-value) models. The progenitor of all structural models is the model of Merton[13], which links the default of a firm to the relationship between its assets and the liabilities at the end of a given time period $[0,T]$. More precisely, in a structural credit portfolio model the i-th counterparty defaults if its asset return (or ability-to-pay) variable A_i falls below a default threshold D_i. In the classical Merton model, A_i is assumed to be a normally distributed random variable and the default threshold is a real number.

The portfolio loss of a portfolio with d counterparties is given by

$$L := \sum_{i=1}^{d} l_i \cdot \mathbf{1}_{\{A_i \le D_i\}}, \tag{1}$$

where l_i is the loss-at-default of the i-th counterparty and $\mathbf{1}_{\{A_i \le D_i\}}$ is a random variable that takes value 1 if counterparty i defaults and 0 otherwise.

To reflect risk concentrations, each A_i is decomposed into a sum of systematic factors $X_1, \ldots, X_m$, which are often identified with geographic regions or industries, and a counterparty-specific factor ε_i, that is,

$$A_i = \sqrt{R_i^2} \sum_{j=1}^{m} w_{ij} X_j + \sqrt{1 - R_i^2}\varepsilon_i, \quad i = 1, \ldots, d. \tag{2}$$

It is usually assumed that the vector of systematic factors $(X_1, \ldots, X_m)$ follows an m-dimensional normal distribution with mean $\mathbf{0} = (0, \ldots, 0)$. The systematic weights $w_{i1}, \ldots, w_{im} \in \mathbb{R}$ determine the impact of each systematic factor on the ability-to-pay variable A_i. The systematic weights are scaled such that the systematic component $Z_i := \sum_{j=1}^{m} w_{ij} X_j$ is a standardized normally distributed variable, i.e., Z_i has mean 0 and variance 1. The idiosyncratic factors $\varepsilon_1, \ldots, \varepsilon_d$ are standardized normally distributed variables, which are independent of each other as well as independent of the systematic factors. Each R_i^2 is an element of the unit interval $[0,1]$. It determines the impact of the systematic component on A_i and therefore the correlation between A_i and Z_i: it immediately follows from (2) that $R_i^2 = \mathrm{Corr}(A_i, Z_i)^2$.

Portfolio risk is quantified by risk measures on the portfolio loss distribution (1). The *expected loss* of the credit portfolio is used for specifying credit reserves. It is defined as the mean of L,

$$\mathbb{E}(L) = \sum_{i=1}^{d} l_i \cdot p_i,$$

where $p_i = \mathbf{P}(A_i \le D_i)$ denotes the default probability of the i-th counterparty. Capital requirements for covering unexpected losses are typically derived from the *value-at-risk* $\mathrm{VaR}_\alpha(L)$ for a predefined probability $\alpha \in (0,1)$, where $\mathrm{VaR}_\alpha(L)$ specifies the loss amount that is not exceeded with probability α. For determining capital requirements, many banks apply the risk measure *economic capital*, which is just the difference between value-at-risk and expected loss. The critical quantities entering the risk measures are the default probabilities and the risk concentrations of the default variables $\mathbf{1}_{\{A_i \le D_i\}}$, either specified by the default correlations $\mathrm{Corr}(\mathbf{1}_{\{A_i \le D_i\}}, \mathbf{1}_{\{A_j \le D_j\}})$ or by the asset correlations $\mathrm{Corr}(A_i, A_j)$.

2.2 Distribution of model variables

The standard approach in credit risk management is to model the risk factors and ability-to-pay variables through a joint multi-variate normal (aka Gaussian) distribution. Since the purpose of this paper is to analyse the impact of stress scenarios under different distribution assumptions we use a more general model which encompasses many frequently used light-tailed and heavy-tailed distributions.

In order to specify a more flexible dependence structure than normal distributions we introduce an additional random variable W, the so-called *mixing variable*, which is strictly positive and independent of the systematic and idiosyncratic factors. The definition of the ability-to-pay variables is generalized to

$$A_i = W\Big(\sqrt{R_i^2}\sum_{j=1}^{m} w_{ij}X_j + \sqrt{1-R_i^2}\varepsilon_i\Big) \tag{3}$$

$$= \sqrt{R_i^2}\sum_{j=1}^{m} w_{ij}WX_j + \sqrt{1-R_i^2}W\varepsilon_i \tag{4}$$

and the systematic and idiosyncratic risk factors now have the form WX_j and $W\varepsilon_i$ respectively. The ability-to-pay variables and risk factors specified in this way follow a so-called multivariate *normal variance mixture (NVM) distribution*. The most important distribution classes covered in this general model are the multivariate normal distribution, in which case the variable W equals 1, and the multivariate Student-t distribution, where W^2 follows an inverse gamma distribution. The Student-t distribution allows for more extreme events than the normal distribution and is therefore a commonly used alternative in financial modelling. Compared to the normal distribution, it takes one additional parameter, the so-called *degrees of freedom*, denoted by ν, that controls the heaviness of the tails. For more details we refer to McNeil *et al.*[12]

In general, the tail behaviour of the risk factor W determines the so-called *heaviness* of the tails of the A_i: If the tail function $\mathbf{P}(W \geq x)$ follows a power law, e.g. $\mathbf{P}(W \geq x) \approx x^{-\nu}$ for a $\nu > 0$ and large x, then the ability-to-pay variables are said to have heavy tails. If W is bounded or its tail function decays exponentially, e.g. $\mathbf{P}(W \geq x) \approx e^{-x}$ for large x, then $A_1, \ldots, A_d$ are light-tailed.[1] The normally distributed model and the Student-t distributed model are examples of light-tailed and heavy-tailed models, respectively.

2.3 Stress testing in credit portfolio models

In a stress test, credit portfolios are typically evaluated under the assumption of adverse economic conditions. A natural way for implementing stress tests in portfolio models is to translate the stress scenario into constraints on risk factors. In our setup, the constraints are formalised by truncating risk factor variables, that is, by conditioning on the range of values that the risk factors may attain. For example, consider an economic stress scenario that is triggered by a downturn in the German economy. This scenario is implemented in the model by truncating the systematic risk factor for Germany. The severity of the downturn scenario is reflected through the truncation threshold $C \in \mathbb{R}$, so that a lower threshold implies more severe stress. Note that the response of the peripheral (or unstressed) risk factors is specified by the dependence structure of the model.

For the sake of simplicity, it will always be assumed that the first risk factor WX_1 is truncated. We denote this factor by $V := WX_1$.

[1] The precise definition is based on the theory of regular variations, see McNeil *et al.*[12] Heavy-tailed models correspond to a regularly varying tail function of W, whereas a model is light-tailed if W is bounded or its tail function is rapidly varying.

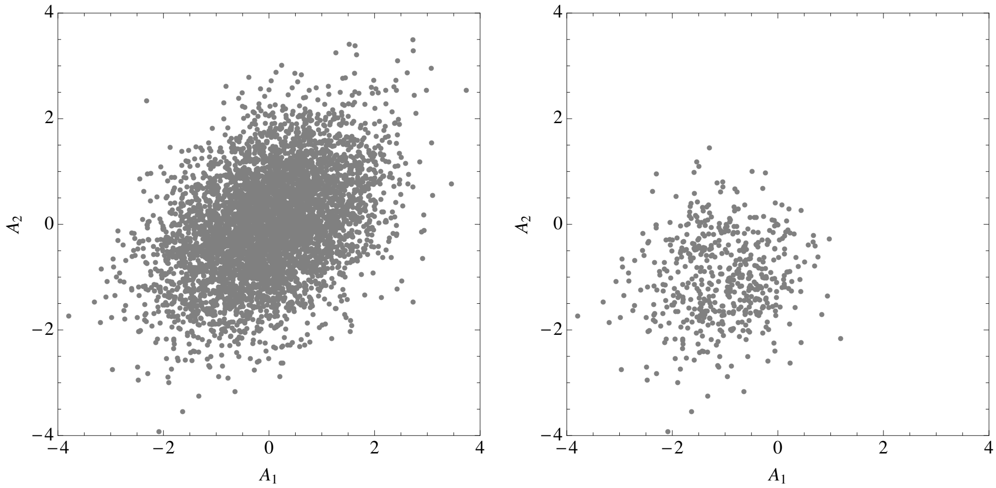

Figure 1: Left: Simulated normally distributed asset returns A_1 and A_2. The assets are correlated with 0.4; each asset is correlated to the joint driving risk factor V with correlation 0.6. Right: Samples conditional on $V \leq -1.28$ which corresponds to a stress event with probability 10%; the correlation of the sample is 0.1, which is far smaller than the original correlation of 0.4.

Figure 1 illustrates the impact of truncating a risk factor: The left plot shows a scatter plot of 5000 simulated samples representing asset returns of two assets, A_1 and A_2. The asset returns are jointly standard normally distributed, and with a correlation of $\rho_{12} = 0.4$. All simulated scenarios are relevant in the unstressed model. In the right plot only those scenarios are shown where the stressed risk factor V does not exceed a threshold C, where C is chosen such that the stress probability $\mathbf{P}(V \leq C)$ equals 10%. As a consequence, only approximately 500 of the 5000 scenarios are considered under stress. Since the A_i and V have a positive correlation of 0.6 the average asset return in the stressed model is negative, which results in a higher number of defaults. It can also be observed that the asset correlation of 0.4 is significantly reduced under stress, i.e., the correlation of A_1 and A_2 drops to 0.1. The impact of stress on asset correlation will be further analysed in the next section.

The stress testing technique described above is commonly used in credit risk management and capital management of financial institutions, see e.g. Bonti *et al.*[3]; Duellmann and Erdelmeier[7]; Kalkbrener and Packham[10]. It has a number of advantages, most importantly:

1. Stress scenarios are implemented in a way that is consistent with the existing quantitative framework. This implies that the relationships between (unrestricted) risk factors remain intact and the experience gained in the day-to-day use of the model can be utilized in the interpretation of stress testing results.

2. The probability of each stress scenario, e.g. the probability that the risk factors satisfy all the constraints under non-stress conditions, can be easily calculated in the statistical model. This is a good indicator for the severity of a stress scenario.

For further stress testing methods, we refer to e.g. Kupiec[11], Berkowitz[2] and Breuer *et al.*[4] Hopper[9] provides case studies on different aspects of stress testing.

The objective of this paper is to analyse default probabilities, asset and default correlations and risk measures under stress. Using techniques from *extreme value theory (EVT)*, we obtain closed formulas when taking the stress level to infinity, i.e., when the truncation threshold C converges to $-\infty$.

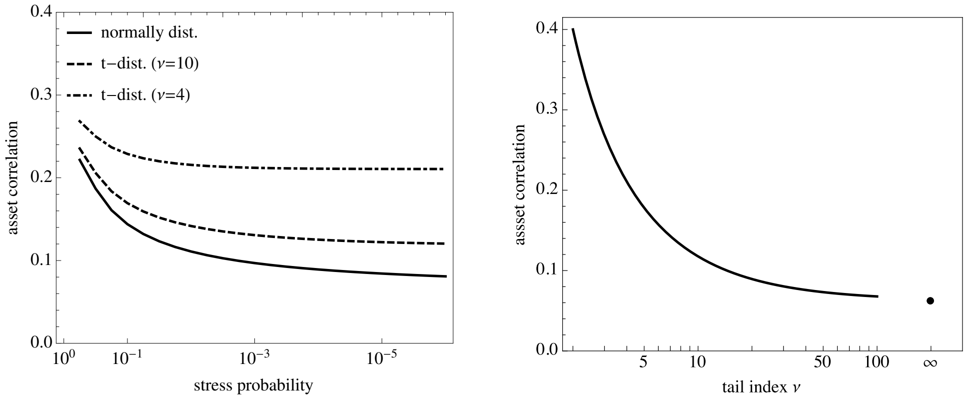

Figure 2: Left: Stressed asset correlation for different distribution assumptions as a function of the stress probability. Right: Stressed asset correlation as a function of the tail index when the stress event is taken to the limit $-\infty$. Correlations are as in Figure 1.

3 Asset correlations under stress

The correlations investigated in Figure 1 are the so-called *asset correlations*, that is, the correlations between the asset returns of the counterparties. For asset returns A_i and A_j we denote their (unconditional) correlation by ρ_{ij}; the correlation of A_i with risk factor V will be denoted by ρ_i.

It turns out that asset correlations are less sensitive to stress in heavy-tailed models than in light-tailed models. For illustration, we compare the asset correlation $\rho_{12} = \mathrm{Corr}(A_1, A_2)$ under stress in the light-tailed normally distributed model and in the heavy-tailed t-distributed model using the same correlation assumptions as in Figure 1. The left graph of Figure 2 shows stressed asset correlations, where instead of the stress level C, stress is expressed by stress probabilities, which are just the probabilities associated with the stress event, $\mathbf{P}(V \leq C)$. For instance, values at 10^{-1} correspond to a stress scenario with probability 10%. Stressed asset correlations are shown for normally distributed and t-distributed assets with degrees of freedom $\nu = 10$ and $\nu = 4$.

Stressed asset correlations may be either greater or smaller than the unconditional asset correlation depending largely on the correlations between the risk factor and the respective asset returns. As illustrated in Figure 2, when the assets in question are sufficiently correlated with the risk factor, the stressed correlation is typically smaller than the unstressed correlation. Loosely speaking, in such a case systematic risk is reduced by conditioning on the risk factor, whereas unsystematic risk remains.

The stressed correlations in the left graph of Figure 2 are calculated with analytic formulas derived in Kalkbrener and Packham [10]. For normally distributed asset returns A_i, A_j their asset correlations conditional on stress level C are given by

$$\mathrm{Corr}^C(A_i, A_j) = \frac{\rho_i\,\rho_j\,\mathrm{Var}^C(V) + \rho_{ij} - \rho_i\,\rho_j}{\sqrt{(\rho_i^2\,\mathrm{Var}^C(V) + 1 - \rho_i^2)(\rho_j^2\,\mathrm{Var}^C(V) + 1 - \rho_j^2)}}, \tag{5}$$

with

$$\mathrm{Var}^C(V) = 1 - \frac{C\,\phi(C)}{\mathrm{N}(C)} - \frac{(\phi(C))^2}{(\mathrm{N}(C))^2},$$

where ϕ denotes the standard normal density function and N denotes the standard normal distribution function. A corresponding, but more involved, formula is also derived for the Student t-distribution.

The severity of the stress is increased by setting the stress level C to higher negative values, or equivalently, reducing the probability of the stress scenario specified by $\mathbf{P}(V \leq C)$. By letting $\mathbf{P}(V \leq C)$ converge to 0, e.g. by moving to the right in the left graph of Figure 2, we arrive at the asymptotic limit, which is of particular importance for understanding the model behaviour under stress. The right-hand side graph of Figure 2 shows the asymptotic limit of stressed asset correlations for t-distributed assets with different values for ν, where $\nu = \infty$ corresponds to the normally distributed case. The asymptotic analysis confirms the higher sensitivity of light-tailed asset variables under stress.

We have also derived concrete formulas for the asymptotic case, see Kalkbrener and Packham [10]. These formulas hold in the more general setup of normal variance mixture models. For heavy-tailed NVM models the asymptotic limit of the stressed correlation of A_i and A_j equals

$$\frac{\rho_i \, \rho_j + (\rho_{ij} - \rho_i \, \rho_j)\,(\nu - 1)}{\sqrt{(\rho_i^2 + (1 - \rho_i^2)\,(\alpha - 1))\,(\rho_j^2 + (1 - \rho_j^2)\,(\nu - 1))}}, \quad \nu > 2, \tag{6}$$

if the risk factor is stressed asymptotically, i.e., if V is truncated at a threshold C, and C converges to $-\infty$. The parameter ν specifies the tail index of the asset returns and the risk factor in the heavy-tailed case and corresponds just to the degrees of freedom defined for t-distributions. The case when the variables are light-tailed corresponds to the limit as $\nu \to \infty$, in which case the asymptotic limit of the conditional correlation between A_i and A_j is

$$\frac{\rho_{ij} - \rho_i \, \rho_j}{\sqrt{(1 - \rho_i^2)(1 - \rho_j^2)}}. \tag{7}$$

Finally, note that the analysis in this section is not restricted to credit portfolio models but holds for any portfolio model with asset variables and risk factors that follow a normal variance mixture distribution.

4 Default probabilities and default correlations under stress

The credit-specific quantities entering credit portfolio models are the default probabilities and default correlations. Just as for asset correlations, their asymptotic behaviour depends on whether the credit portfolio model follows a light- or heavy-tailed NVM distribution. In the light-tailed case, default probabilities converge to 1 under extreme stress and default correlations converge to 0.[2] In other words, default of the entire portfolio becomes a sure event under extreme stress and correlations between default indicators become irrelevant.

In contrast, asymptotic default probabilities and asymptotic default correlations are in $(0, 1)$ in the heavy-tailed case. Both quantities depend on the tail index ν and can be expressed in terms of the Student t-distribution function. More specifically, the asymptotic default probability under stress for a model with tail index ν is given by [1;14]

$$\lim_{C \to -\infty} \mathbf{P}(A_1 \leq D_1 | V \leq C) = t_{\nu+1}\left(\frac{\sqrt{\nu + 1}\,\rho_1}{\sqrt{1 - \rho_1^2}}\right) \in [1/2, 1), \tag{8}$$

where t_ν is the distribution function of the Student-t distribution with parameter ν. A formula for bivariate default probabilities – albeit more involved – and an integral representation for

[2] In this section, we assume that the unconditional correlations between asset returns $A_1, \ldots, A_d$ and the risk factor V are positive and less than 1, i.e., $\rho_i, \rho_{ij} \in (0, 1)$ for $i, j \in \{1, \ldots, d\}$.

multivariate default probabilities that can be calculated numerically, are derived in Packham *et al.*[14]

In all models – whether heavy-tailed or light-tailed – the asymptotic limit of stressed default probabilities and default correlations does not depend on the unstressed default probabilities. For the heavy-tailed case, the tail index and the unstressed correlations enter the asymptotic results.

In summary, the impact of stress on the asymptotic limit of default probabilities and correlations is greater in light-tailed models than in heavy-tailed models. This is a remarkable observation since light-tailed models, in particular normally distributed models, are usually considered less sensitive to extreme stress than heavy-tailed models: a popular measure in finance to assess the ability of a bivariate distribution to generate joint extreme events – the tail dependence – is zero in light-tailed models, whereas it is a positive number in heavy-tailed models. In order to better understand this phenomenon we now compare the behaviour of limiting default probabilities to tail dependence.

The tail dependence, or more precisely, the *coefficient of (lower) tail dependence* of the identically distributed variables V and A_1 is defined as[3]

$$\lambda_l(V, A_1) := \lim_{C \to -\infty} \mathbf{P}(A_1 \leq C | V \leq C). \tag{9}$$

Hence, the tail dependence of V and A_1 measures the probability $\mathbf{P}(A_1 \leq C)$ conditional on the event $\{V \leq C\}$ for stress levels C converging to $-\infty$. If the NVM distributed random variables V, A_1 are heavy-tailed with tail index ν, the tail dependence coefficient is given by[12]

$$\lambda_l(V, A_1) = 2t_{\nu+1}\left(-\sqrt{\frac{(\nu+1)(1-\rho_1)}{1+\rho_1}}\right).$$

It follows that the tail dependence is strictly positive for heavy-tailed models, provided that $\rho_1 > -1$. For light-tailed NVM distributions, the tail dependence is zero. This includes, of course, the normal distribution, which is still the de-facto standard for modelling risk factors and asset log-returns in structural credit portfolio models, such as CreditMetrics™ [8] and Moody's KMV Portfolio Manager™ [6].

The zero tail dependence is in contrast to the asymptotic default probability in the light-tailed case, where default is a sure event. Similarly, tail dependence and asymptotically stressed default probabilities disagree in the heavy-tailed case. The left graph of Figure 3 illustrates the difference between tail dependence and asymptotic stressed PD's as a function of the tail index ν.

To make the relation between tail dependence and asymptotic stressed PD's more precise, we introduce an additional parameter $x \in \mathbb{R}$ and measure the probability $\mathbf{P}(A_1 \leq x \cdot C)$ conditional on the event $\{V \leq C\}$ for stress levels C converging to $-\infty$. More formally, we consider the function

$$\lambda(V, A_1, x) := \lim_{C \to -\infty} \mathbf{P}(A_1 \leq x \cdot C | V \leq C), \quad x \in \mathbb{R},$$

which provides an elegant generalization of both concepts: the tail dependence coefficient of V and A_1 equals $\lambda(V, A_1, 1)$, whereas the asymptotic stressed PD corresponds to $\lambda(V, A_1, 0)$. A closed-form expression for $\lambda(V, A_1, x)$ can be obtained via elementary transformations from Abdous *et al.*[1], see also Packham *et al.*[14] The tail dependence function is illustrated in the right graph of Figure 3.

[3] In the general case, when V and A_1 are not identically distributed, the tail dependence coefficient is defined via quantiles[12].

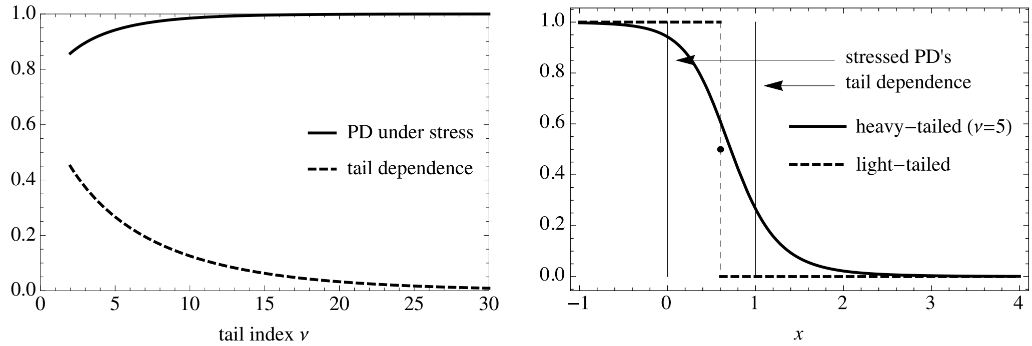

Figure 3: Left: Tail dependence coefficient and asymptotic PD under stress as a function of the tail index ν. Right: Tail dependence function $\lambda(V, A_1, x)$ for light- and heavy-tailed variables; special cases arise at $x = 0$ (stressed PD's) at $x = 1$ (tail dependence). The initial correlation between the ability-to-pay variable and the risk factor is 0.6 in both cases.

The analysis of the function $\lambda(V, A_1, x)$ illustrates the fundamentally different behaviour of the tail dependence coefficient and asymptotic stressed PD's in light-tailed and heavy-tailed credit portfolio models. In the light-tailed case, the asset variable A_1 converges to $-\infty$, more specifically, it is concentrated at $\rho_1 \cdot C$ when $V \leq C$ and $C \to -\infty$. In the heavy-tailed case, however, A_1 does not show the same uniform asymptotic behaviour: $0 < \lambda(V, A_1, x) < 1$ holds for all $x \in \mathbb{R}$ and, in particular, tail dependence as well as stressed default probabilities are in $(0, 1)$.

In summary, this analysis clearly shows that the tail dependence coefficient only provides partial information on a model's ability to produce extreme (joint) events. A more comprehensive picture is given by function $\lambda(V, A_1, x)$, which also explains the observed differences between tail dependence and asymptotic stressed PD's.

So far, our analysis has focused on asymptotic stressed default probabilities. For practical purposes, the model behaviour at smaller and therefore more realistic stress levels is even more important. Hence, we now take a closer look at PD's under stress for various stress levels C and compare them in light- and heavy-tailed models. Figure 4 shows PD's under stress for both normally distributed and t-distributed ($\nu = 3$) models as a function of the stress probabilities. The unconditional correlation between the ability-to-pay variable and the risk factor is 0.6. Despite converging to a value smaller than 1, PD's under stress in the t-distributed model dominate the normally distributed case unless the stress probability is very small: If the unconditional PD is 10%, then for stress probabilities greater than approximately $10^{-3.5}$, the PD under stress in the t-distributed model is greater than the respective PD in the normal model. If the unconditional PD is 1%, then the threshold lies beyond 10^{-8}.

This example shows that for realistic stress tests the impact on PD's is usually greater in heavy-tailed models. Only for rather extreme stress severities, stressed PD's become higher in light-tailed models and eventually converge to 1.

5 Risk measures

The different behaviour of light-tailed and heavy-tailed models has implications on the credit reserves and capital requirements in stress scenarios, as demonstrated by the following stylized example. Consider a homogeneous portfolio consisting of 60 counterparties. Each counterparty has notional and loss-at-default of 1/60 and defaults with a probability of 1%. The asset variables of the counterparties are correlated through one risk factor, with $\rho = 0.4$ the correlation

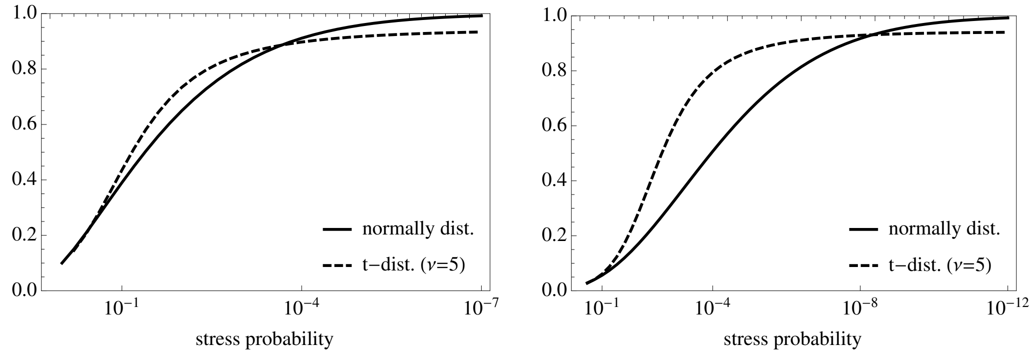

Figure 4: PD's under stress as a function of the stress probability. Models considered are the normal distribution and the t-distribution with parameter $\nu = 5$. Correlations are 0.6. Left: unconditional PD is 0.1. Right: unconditional PD is 0.01.

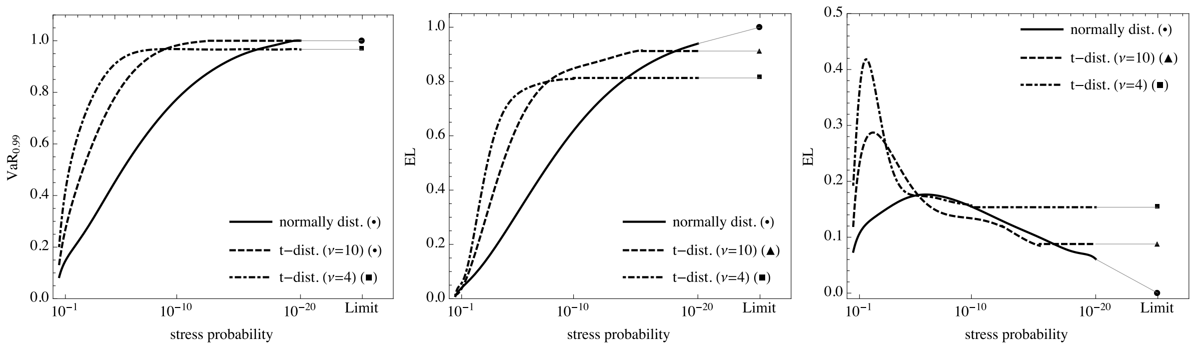

Figure 5: Risk measures for portfolio consisting of 60 homogeneous counterparties, each with a PD of 1%. Left: Value-at-risk at 99% confidence level; middle: Expected loss; right: Economic capital.

between any one counterparty and the risk factor. This implies that the counterparties are correlated with $\rho^2 = 0.16$.

Figure 5 shows the value-at-risk, the expected loss and the resulting economic capital for the portfolio under different distribution assumptions, i.e., under a normal distribution and t-distributions with $\nu = 4$ and $\nu = 10$, and for different stress levels. As before, the stress level C is translated into a stress probability, which denotes the probability that a certain stress event occurs. The left graph shows the 99%-value-at-risk of the portfolio. Despite being lower under moderate stress, the VaR in a normally distributed model converges to 1, whereas the VaR for a very heavy-tailed model (t-distribution with $\nu = 4$) converges to a number strictly smaller than 1, see Packham *et al.*[14] for the calculation of the asymptotic results. When comparing the two t-distributed models, the more heavy-tailed model with $\nu = 4$ has higher risk for moderate stress levels, but lower risk for less probable stress events.

Similar observations hold for the expected loss (middle graph). The expected loss under stress corresponds just to the probability of default under stress, since the recovery rate is 0 in this example. The asymptotic results, Equation (8), confirm that the EL converges to 1 in the light-tailed case, whereas it converges to a number strictly smaller than 1 in the heavy-tailed cases. Finally, economic capital converges to zero for normally distributed models and to a number strictly greater than zero for heavy-tailed models (a confidence level of 99% for economic capital may not be realistic in practice, but serves well to illustrate some key characteristics of the stressed portfolios). Because stress has different impact on value-at-risk and expected loss, economic capital is not monotone, but increases under moderate stress and decreases for greater stress levels.

To conclude, in light-tailed models, extreme stress scenarios tend to heavily increase the credit reserves specified by the expected loss whereas economic capital, which defines capital requirements, converges to 0. The impact of extreme stress on expected loss and economic capital is more balanced in heavy-tailed models, whose asymptotic limit retains a richer dependence structure.

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