# Initial complexes of prime ideals

Michael Kalkbrener
Department of Mathematics
ETH-Zürich
CH-8092 Zürich, Switzerland

Bernd Sturmfels
Department of Mathematics
Cornell University
Ithaca, NY 14853, USA

**Abstract**

The simplicial complex defined by any initial ideal of a prime ideal is pure and strongly connected. This proves a conjecture of Kredel and Weispfenning [12, p.234].

## 1 Introduction

A main theme in Gröbner bases theory is to compute invariants of an ideal $I \subset K[x_1, \ldots, x_n]$ by studying its initial monomial ideal with respect to some term order $\prec$. A fundamental such invariant is the *Krull dimension* $dim(I)$ of the residue ring $K[x_1, \ldots, x_n]/I$.

Throughout this paper $K$ denotes an arbitrary field. Let us translate the definition of dimension given by Gröbner [10, p. 39] into the language of algebraic combinatorics (see [15], [16]). A subset $X$ of $\{x_1, \ldots, x_n\}$ is *independent* modulo $I$ if and only if $K[X] \cap I = \{0\}$. The *independence complex* $\Delta(I)$ of $I$ is the simplicial complex

$$\Delta(I) := \{X \subseteq \{x_1, \ldots, x_n\} : X \text{ is independent modulo } I\}.$$

The Krull dimension $dim(I)$ equals $dim(\Delta(I)) + 1$, the maximal cardinality of a face in $\Delta(I)$. Every simplicial complex on $\{x_1, \ldots, x_n\}$ arises in this way: it is the independence complex of the *Stanley-Reisner ideal* generated by its non-faces (see [15]).

For any prime ideal $P$ the complex $\Delta(P)$ is a *matroid complex* (see [2] and [15, section 7]). This means that $\Delta(P)$ is *pure* (i.e., all maximal faces have the same dimension), and $\Delta(P)$ satisfies the Steinitz exchange property [12, Lemma 1.2]. In particular, $\Delta(P)$ is *strongly connected*, i.e., for any two maximal faces $X, X'$ there exists a sequence of maximal faces $X = X_0, X_1, X_2, \ldots, X_k = X'$ such that $|X_i \setminus X_{i-1}| = |X_{i-1} \setminus X_i| = 1$ for $i = 1, \ldots, k$.

Let $\prec$ be any term order on $K[x_1, \ldots, x_n]$. The initial monomial of a polynomial $f \in K[x_1, \ldots, x_n]$ is denoted by $init_\prec(f)$. The *initial ideal* $init_\prec(I)$ of an ideal $I$ in $K[x_1, \ldots, x_n]$ is the ideal generated by the initials of the polynomials in $I$. We define the *initial complex* of $I$ as $\Delta_\prec(I) := \Delta(init_\prec(I))$, the independence complex of the initial ideal. Note that $\Delta_\prec(I) \subseteq \Delta(I)$. The results in [4] and [12] imply that

$$dim\,(\Delta_\prec(I)) \quad = \quad dim\,(\Delta(I)) \quad = \quad dim(I) - 1.$$

Thus to compute $dim(I)$ it suffices to find a face of maximal cardinality in $\Delta_\prec(I)$. Kredel and Weispfenning [12] showed how to perform this task using Buchberger's Gröbner bases algorithm [3]. For the case of a prime ideal $P$ they conjectured that $\Delta_\prec(P)$ is pure [12, p.234]. In this paper we prove the conjecture of Kredel and Weispfenning.

**Theorem 1** *Let $P$ be a prime ideal in $K[x_1,\ldots,x_n]$ and $\prec$ any term order. Then the initial complex $\Delta_\prec(P)$ is pure of dimension $dim(P)-1$ and strongly connected.*

This shows that the greedy algorithm for choosing a maximal face [12, Corollary 2.2] determines the correct dimension, using any $\prec$. We note that a pure zero-dimensional complex consisting of $n \geq 2$ points is strongly connected but not connected. For instance, the prime ideal $P = \langle x_1x_2 - 1\rangle$ in $K[x_1,x_2]$ has the initial complex $\Delta_\prec(P) = \{\{x_1\},\{x_2\},\emptyset\}$.

This research was conducted during the 1991-92 stay of the first author at the Mathematical Sciences Institute of Cornell University, which is supported by the U.S. Army Research Office. The second author is partly supported by the National Science Foundation and the Alfred P. Sloane Foundation. We thank David Eisenbud for helpful comments.

## 2 Proof of Theorem 1

We first summarize some basic facts about independence complexes. The proof of Lemma 1 is immediate from the definitions.

**Lemma 1** *Let $I$ and $J$ be ideals in $K[x_1,\ldots,x_n]$.*
**(a)** *If $I \subseteq J$ then $\Delta(I) \supseteq \Delta(J)$.*
**(b)** *$\Delta(Rad(I)) = \Delta(I)$, where $Rad(I)$ denotes the radical of $I$.*
**(c)** *$\Delta(I \cap J) = \Delta(I \cdot J) = \Delta(I) \cup \Delta(J)$.*
**(d)** *Let $P_1, P_2, \ldots, P_r$ be the associated prime ideals of $I$. Then*

$$\Delta(I) \quad = \quad \Delta(P_1) \cup \Delta(P_2) \cup \ldots \cup \Delta(P_r).$$

We recall the representation of a term order $\prec$ by a non-negative weight vector $\omega = (\omega_1,\ldots,\omega_n) \in \mathbf{N}^n$. Let $f = f(x_1,\ldots,x_n)$ be any polynomial in $K[x_1,\ldots,x_n]$. We consider $f(t^{\omega_1}x_1,\ldots,t^{\omega_n}x_n)$ as a univariate polynomial in $t$. Its leading coefficient $init_\omega(f)$ is a polynomial in $K[x_1,\ldots,x_n]$. We call it the *initial form* of $f$ with respect to $\omega$.

Fix an ideal $I \subset K[x_1,\ldots,x_n]$. For any $\omega \in \mathbf{N}^n$ we define the *initial ideal* $init_\omega(I)$ to be the ideal generated by $\{init_\omega(f) \,:\, f \in I\}$. For sufficiently generic $\omega$ the initial ideal $init_\omega(I)$ is a monomial ideal. In this case there exists a term order $\prec$ such that

$$init_\prec(I) \quad = \quad init_\omega(I),$$

and we say that the vector $\omega$ *represents* the term order $\prec$ for $I$. The following lemma is a standard result in Gröbner bases theory (see [13, Proposition 2.3]).

**Lemma 2** *Let $G$ be the reduced Gröbner basis for $I$ with respect to $\prec$. A vector $\omega \in \mathbf{N}^n$ represents $\prec$ for $I$ if and only if $init_\omega(g) = init_\prec(g)$ for all $g \in G$.*

This implies that for a fixed ideal $I$ every term order $\prec$ can be represented by some $\omega \in \mathbf{N}^n$. We will use this result to prove the following lemma.

**Lemma 3** *Let $P$ be a $d$-dimensional prime ideal in $K[x_1,\dots,x_n]$ and $\prec$ any term order. Then there exists a $(d+1)$-dimensional prime ideal $P'$ in $K[x_1,\dots,x_n,x_{n+1}]$ such that*

$$init_\prec(P) + \langle x_{n+1}\rangle \quad = \quad P' + \langle x_{n+1}\rangle \qquad\qquad \textit{(as ideals in } K[x_1,\dots,x_{n+1}]\textit{)}. \tag{1}$$

**Proof:** Let $(s_1,\dots,s_n)$ denote the generic point of $P$ and let $t$ be a new variable which is algebraically independent from $\{s_1,\dots,s_n\}$. Suppose that $\omega = (\omega_1,\dots,\omega_n) \in \mathbf{N}^n$ represents $\prec$ for $P$. Let $P'$ be the prime ideal in $K[x_1,\dots,x_n,x_{n+1}]$ which has the generic point $(s_1t^{\omega_1},\dots,s_nt^{\omega_n},t)$.

Since $\omega$ represents $\prec$ for $P$, it suffices to prove that

$$init_\omega(P) + \langle x_{n+1}\rangle \quad = \quad P' + \langle x_{n+1}\rangle. \tag{2}$$

To prove the inclusion "$\subseteq$" in (2), we consider any $g \in P$ and we define

$$g' \quad := \quad g\,(\frac{x_1}{x_{n+1}^{\omega_1}}, \dots , \frac{x_n}{x_{n+1}^{\omega_n}}) \cdot x_{n+1}^m,$$

where $m$ is the smallest non-negative integer such that $g' \in K[x_1,\dots,x_n,x_{n+1}]$. It follows from the definition of $init_\omega(g)$ that $g'$ can be written in the form $g' = init_\omega(g) + r$, where $r \in K[x_1,\dots,x_{n+1}]$ is divisible by $x_{n+1}$. From

$$g'(s_1t^{\omega_1},\dots,s_nt^{\omega_n},t) \quad = \quad g(s_1,\dots,s_n)\cdot t^m \quad = \quad 0$$

we conclude that $g' = init_\omega(g) + r$ is in $P'$. Therefore $init_\omega(g) \in P' + \langle x_{n+1}\rangle$.

To prove the reverse inclusion in (2), we consider any $f \in P'$. Writing $f = f_m x_{n+1}^m + \dots + f_1x_{n+1} + f_0$ as a polynomial in $x_{n+1}$ with coefficients $f_i$ in $K[x_1,\dots,x_n]$, we need to show that $f_0$ lies in the initial ideal $init_\omega(P)$. We can assume that $f_0$ is unequal to 0.

We define $f' := f(x_1x_{n+1}^{\omega_1},\dots,x_nx_{n+1}^{\omega_n},x_{n+1})$, and we note that

$$f'(s_1,\dots,s_n,t) \quad = \quad f(s_1t^{\omega_1},\dots,s_nt^{\omega_n},t) \quad = \quad 0.$$

Since $t$ is algebraically independent of $\{s_1,\dots,s_n\}$, the polynomial $f'$ can be written in the form $f' = p_rx_{n+1}^r + \dots + p_1x_{n+1} + p_0$, where $p_0,\dots,p_r$ are elements of $P$. We have

$$f = f'(\frac{x_1}{x_{n+1}^{\omega_1}},\dots,\frac{x_n}{x_{n+1}^{\omega_n}},x_{n+1}) = p_r(\frac{x_1}{x_{n+1}^{\omega_1}},\dots,\frac{x_n}{x_{n+1}^{\omega_n}})\cdot x_{n+1}^r + \dots + p_0(\frac{x_1}{x_{n+1}^{\omega_1}},\dots,\frac{x_n}{x_{n+1}^{\omega_n}}).$$

This identity implies that there exist $i_1,\dots,i_k \in \{0,\dots,r\}$ such that $f_0 = init_\omega(p_{i_1}) + \dots + init_\omega(p_{i_k}) \in init_\omega(P)$. $\triangleleft$

**Proof of Theorem 1:** Let $\bar{K}$ denote the algebraic closure of $K$, $d$ the dimension of $P$ and $Y$ the irreducible affine variety in $\bar{K}^{n+1}$ defined by the $(d+1)$-dimensional prime ideal $P'$ as above. Let $Z = Y \cap \{x_{n+1} = 0\}$ denote the variety of $P' + \langle x_{n+1}\rangle$ in $\bar{K}^{n+1}$. Since the linear form $x_{n+1}$ does not vanish identically on $Y$ we obtain from Corollary I.6.1 (p. 59) in [14] that each irreducible component of $Z$ has dimension $d$. By Lemma 3,

$$Z \; = \; \{(a_1,\dots,a_n,0) \in \bar{K}^{n+1} \, : \, (a_1,\dots,a_n) \text{ is in the variety of } init_\prec(P)\}.$$

Therefore each of the associated prime ideals $P_1, \dots, P_r$ of $init_\prec(P)$ has dimension $d$. Since $\Delta(P_i)$ is pure of dimension $d-1$ for every $i \in \{1, \dots, r\}$, Lemma 1(d) implies that $\Delta_\prec(P)$ is pure of dimension $d-1$.

We next show that $\Delta_\prec(P)$ is strongly connected. First suppose that $\omega = (1, 1, \dots, 1)$. In this case the prime ideal $P'$ in Lemma 3 is homogeneous, as can be seen from its generic point $(s_1 t, \dots, s_n t, t)$. Let $\mathbf{P}^n$ denote the projective $n$-space over $\bar{K}$. Let $Y$ be the $d$-dimensional irreducible projective variety defined by $P'$ and $Z$ the $(d-1)$-dimensional projective variety defined by the monomial ideal $init_\prec(P) + \langle x_{n+1} \rangle$.

Theorem 1 clearly holds if $P$ has affine dimension $d = dim(P) \leq 1$. If $d = 0$ then $\Delta_\prec(P) = \{\emptyset\}$. If $d = 1$ then $\Delta_\prec(P)$ is a zero-dimensional simplicial complex and therefore strongly connected.

We now suppose $d \geq 2$. Each of the associated prime ideals $P_1, \dots, P_r$ of $init_\prec(P)$ is generated by a subset $X_i$ of $\{x_1, \dots, x_n\}$ of cardinality $n - d$. Let $L_i$ denote the linear subspace in $\mathbf{P}^n$ defined by $X_i \cup \{x_{n+1}\}$ for $i \in \{1, \dots, r\}$. We fix a generic linear subspace $L$ of codimension $d-2$ in $\mathbf{P}^n$. Then $L \cap L_i$ defines a line in $\mathbf{P}^n$ for every $i \in \{1, \dots, r\}$, and two lines $L \cap L_i$ and $L \cap L_j$ meet if and only if $|X_i \setminus X_j| = 1$ . Therefore our claim that $\Delta_\prec(P)$ is strongly connected is equivalent to the assertion that $L \cap Z$ is a connected curve. By Lemma 3, it equals the intersection of $Y$ with a codimension $d-1$ linear subspace:

$$L \cap Z \quad = \quad (L \cap \{x_{n+1} = 0\}) \cap Y. \tag{3}$$

The (projective) dimensions of these projective varieties satisfy the inequality

$$dim(L \cap \{x_{n+1} = 0\}) + dim(Y) \quad = \quad (n - d + 1) + d \quad = \quad n + 1 \quad > \quad n.$$

Therefore we can apply the general connectedness theorem of Fulton and Hansen stated in [7, Corollary 1] (see also Theorem 4.1 in [8]). We conclude that (3) is a connected curve. This completes the proof of Theorem 1 for the special case $\omega = (1, 1, \dots, 1)$.

The same argument proves a slightly stronger result. Let $\tilde{P}$ be any prime ideal in $K[x_1, \dots, x_n]$ of dimension $d \geq 2$, and let $\tilde{Z}$ be the variety in $\mathbf{P}^{n-1}$ defined by the homogeneous ideal $init_{\mathbf{1}}(\tilde{P})$, where $\mathbf{1}$ denotes $(1, 1, \dots, 1)$. (This need not be a monomial ideal.) Then the intersection $\tilde{Z} \cap L$ with the generic $(d-2)$-flat $L$ is a connected curve in $\mathbf{P}^{n-1}$.

We now complete the proof of Theorem 1 for the general case. Let $(s_1, \dots, s_n)$ be the generic point of $P$, and let $\omega = (\omega_1, \dots, \omega_n)$ be any positive integral weight vector representing $\prec$ for $P$. Let $\tilde{P}$ denote the prime ideal in $K[x_1, \dots, x_n]$ having the generic point $(s_1^{1/\omega_1}, s_2^{1/\omega_2}, \dots, s_n^{1/\omega_n})$. Let $M$ denote the monomial ideal obtained from $init_\prec(P) = init_\omega(P)$ by the substitution $x_1 \mapsto x_1^{\omega_1}, \dots, x_n \mapsto x_n^{\omega_n}$. It is clear that $M \subseteq init_{\mathbf{1}}(\tilde{P})$.

Let $Z$ be the projective variety defined by $M$, and let $\tilde{Z}$ be the projective variety defined by $init_{\mathbf{1}}(\tilde{P})$. We need to show that $Z \cap L$ is a connected curve. By our previous result, we know that $\tilde{Z} \cap L$ is a connected curve. Hence it suffices to prove $Z = \tilde{Z}$, or, equivalently, that the ideals $M$ and $init_{\mathbf{1}}(\tilde{P})$ have the same radical. By the inclusion $M \subseteq init_{\mathbf{1}}(\tilde{P})$, we know that $Rad(M) \subseteq Rad(init_{\mathbf{1}}(\tilde{P}))$.

To prove the reverse inclusion, we suppose there exists $f \in init_{\mathbf{1}}(\tilde{P})$ which does not lie in the monomial ideal $Rad(M)$. Then there is a monomial $m$ occurring in $f$ which is not in $Rad(M)$. We can choose $m$ to be the initial of $f$ with respect to some term order $\prec$.

We next choose $g \in \tilde{P}$ such that $f$ is the initial form of $g$. Let $\eta_i \in \bar{K}$ be a primitive $\omega_i$-th root of unity, for $i = 1, \ldots, n$. We consider the following polynomial

$$G(x_1, x_2 \ldots, x_n) \quad := \quad \prod_{i_1=0}^{\omega_1-1} \prod_{i_2=0}^{\omega_2-1} \cdots \prod_{i_n=0}^{\omega_n-1} g\,(\eta_1^{i_1} x_1, \eta_2^{i_2} x_2, \ldots, \eta_n^{i_n} x_n).$$

Since $G$ is invariant under the transformations $x_i \mapsto \eta_i x_i$, $i = 1, \ldots, n$, it lies in the polynomial subring $K[x_1^{\omega_1}, x_2^{\omega_2}, \ldots, x_n^{\omega_n}]$. Also, $G \in \tilde{P}$. Therefore the initial form of $G$,

$$F(x_1, x_2 \ldots, x_n) \quad := \quad \prod_{i_1=0}^{\omega_1-1} \prod_{i_2=0}^{\omega_2-1} \cdots \prod_{i_n=0}^{\omega_n-1} f\,(\eta_1^{i_1} x_1, \eta_2^{i_2} x_2, \ldots, \eta_n^{i_n} x_n),$$

lies in the monomial ideal $M$, and hence so does each of its monomials.

All factors in the product defining $F$ have the same initial monomial $m$ with respect to $\prec$. Therefore a power of $m$ appears in the expansion of $F$ and thus lies in $M$. This is a contradiction, and the proof of Theorem 1 is complete. $\triangleleft$

## 3 Initial complexes for arbitrary weight vectors

The concept of initial complexes of an ideal $I$ naturally extends from term orders $\prec$ to arbitrary weight vectors $\omega \in \mathbf{N}^n$. We simply define $\Delta_\omega(I) := \Delta(init_\omega(I))$. Our main theorem generalizes to this setting.

**Theorem 2** *Let $P$ be a prime ideal in $K[x_1, \ldots, x_n]$ and $\omega \in \mathbf{N}^n$. Then the initial complex $\Delta_\omega(P)$ is pure of dimension $dim(P) - 1$ and strongly connected.*

Theorem 2 has already been established for vectors $\omega$ which represent term orders. It also holds for $\omega = 0$, the zero vector, because $\Delta_\omega(P) = \Delta(P)$ is a matroid complex [2]. Before giving the proof of Theorem 2, we generalize Lemma 1 to initial complexes.

**Lemma 4** *Let $I$ and $J$ be ideals in $K[x_1, \ldots, x_n]$ and $\omega \in \mathbf{N}^n$ any weight vector.*
**(a)** *If $I \subseteq J$ then $\Delta_\omega(I) \supseteq \Delta_\omega(J)$.*
**(b)** *$\Delta_\omega(Rad(I)) = \Delta_\omega(I)$, where $Rad(I)$ denotes the radical of $I$.*
**(c)** *$\Delta_\omega(I \cap J) \;=\; \Delta_\omega(I \cdot J) \;=\; \Delta_\omega(I) \cup \Delta_\omega(J)$.*
**(d)** *Let $P_1, P_2, \ldots, P_r$ be the associated prime ideals of $I$. Then*

$$\Delta_\omega(I) \quad = \quad \Delta_\omega(P_1) \cup \Delta_\omega(P_2) \cup \ldots \cup \Delta_\omega(P_r).$$

**Proof:** We prove Lemma 4 by applying Lemma 1 to initial ideals. Statement (a) follows from $init_\omega(I) \subseteq init_\omega(J)$, and (b) follows from $Rad(init_\omega(I)) = Rad(init_\omega(Rad(I)))$. The left equation in (c) follows from part (b) and $Rad(I \cdot J) = Rad(I \cap J)$. The inclusion $\Delta_\omega(I \cdot J) \subseteq \Delta_\omega(I) \cup \Delta_\omega(J)$ follows from $init_\omega(I) \cdot init_\omega(J) \subseteq init_\omega(I \cdot J)$, while the reverse inclusion follows from (a). Part (d) is a direct consequence of (a),(b),(c). $\triangleleft$

Our next lemma states that every independence complex is the union of its initial complexes.

**Lemma 5** *Let $I \subset K[x_1, \ldots, x_n]$ be any ideal. Then*

$$\Delta(I) \quad = \quad \bigcup_{\omega \in \mathbf{N}^n} \Delta_\omega(I) \quad = \quad \bigcup_{\prec \text{ term order}} \Delta_\prec(I). \tag{4}$$

**Proof:** Each initial complex $\Delta_\omega(I)$ is contained in the independence complex $\Delta(I)$, and therefore $\Delta(I) \supseteq \bigcup_\omega \Delta_\omega(I)$. The inclusion $\bigcup_\omega \Delta_\omega(I) \supseteq \bigcup_\prec \Delta_\prec(I)$ follows from the fact that every term order $\prec$ is represented by some weight vector $\omega$.

We complete the proof by showing the following assertion: For every face $X$ of $\Delta(I)$ there exists a term order $\prec$ such that $X$ lies in $\Delta_\prec(I)$. We choose an ordering of the variables $x_1, \ldots, x_n$ such that each variable in $X$ is smaller than each variable not in $X$. Let $\prec$ be the purely lexicographic term order on $K[x_1, \ldots, x_n]$ induced from this variable ordering. Our hypothesis $X \in \Delta(I)$ states that $I \cap K[X] = \{0\}$. Because of the definition of $\prec$ this implies $init_\prec(I) \cap K[X] = \{0\}$ and hence $X \in \Delta_\prec(I)$. $\triangleleft$

The preceding proof shows that equation (4) remains valid if the right hand union is restricted to the $n\,!$ purely lexicographic term orders.

We next derive a local refinement of Lemma 5. Let $I$ be any ideal in $K[x_1, \ldots, x_n]$ and $\omega \in \mathbf{N}^n$ any weight vector. There exists an open neighborhood $U_\omega$ of $\omega$ in $\mathbf{Q}^n$ such that

$$\text{for all } \omega' \in \mathbf{N}^n \text{ and all } m \in \mathbf{N} : \quad (1/m) \cdot \omega' \in U_\omega \text{ implies } init_{\omega'}(init_\omega(I)) = init_{\omega'}(I). \tag{5}$$

We say that a term order $\prec$ *refines* $\omega \in \mathbf{N}^n$ for an ideal $I$ if the following condition holds: For every open neighborhood $U$ of $\omega$ in $\mathbf{Q}^n$ there exist an $\omega' \in \mathbf{N}^n$ which represents $\prec$ for $I$ and an $m \in \mathbf{N}$ such that $(1/m) \cdot \omega' \in U$. If $\prec$ refines $\omega$ for $I$ then

$$init_\prec(I) \quad = \quad init_\prec(init_\omega(I)). \tag{6}$$

**Lemma 6** *Let $I \subset K[x_1, \ldots, x_n]$ be any ideal and $\omega \in \mathbf{N}^n$. Then $\Delta_\omega(I)$ is the union of the initial complexes $\Delta_\prec(I)$, where $\prec$ runs over all term orders which refine $\omega$.*

**Proof:** Applying Lemma 5 to $init_\omega(I)$, we find that $\Delta_\omega(I) = \cup_\prec \Delta_\prec(init_\omega(I))$, where $\prec$ runs over all term orders. Hence it suffices to prove the following claim: For any term order $\prec$, there exists a term order $\prec'$ refining $\omega$ such that $init_\prec(init_\omega(I)) = init_{\prec'}(I)$. To see this, let $v \in \mathbf{N}^n$ represent $\prec$ for both $I$ and $init_\omega(I)$. Because of (5) we can choose $m \gg 0$ such that $init_{\omega'}(I) = init_{\omega'}(init_\omega(I))$ for $\omega' := m \cdot \omega + v$. Obviously,

$$init_\prec(init_\omega(I)) \; = \; init_v(init_\omega(I)) \; = \; init_{\omega'}(init_\omega(I)) \; = \; init_{\omega'}(I). \tag{7}$$

Since $init_{\omega'}(I)$ is a monomial ideal, $\omega'$ represents a term order $\prec'$ for $I$. Together with (7) we obtain $init_\prec(init_\omega(I)) \; = \; init_{\prec'}(I)$. Our choices imply that $\prec'$ refines $\omega$ for $I$. $\triangleleft$

**Proof of Theorem 2:** The initial complex $\Delta_\omega(P)$ is pure of dimension $dim(P) - 1$ because it is a union of pure complexes, by Theorem 1 and Lemma 6.

To show that $\Delta_\omega(P)$ is strongly connected, we consider any two maximal faces $X$ and $Y$ of $\Delta_\omega(P)$. Let $P_1, P_2, \ldots, P_r$ be the associated prime ideals of $init_\omega(P)$. By Lemma 1(d), there exist indices $i$ and $j$ such that $X \in \Delta(P_i)$ and $Y \in \Delta(P_j)$. Both $\Delta(P_i)$ and $\Delta(P_j)$ are matroid complexes and hence strongly connected.

Let $\prec$ be any term order which refines $\omega$ for $P$. Then $\Delta_\prec(P_i) \subseteq \Delta(P_i)$ and $\Delta_\prec(P_j) \subseteq \Delta(P_j)$, both pure subcomplexes of the same dimension. We can connect $X$ by a chain of maximal faces in $\Delta(P_i)$ to a maximal face $X'$ of $\Delta_\prec(P_i)$. Similarly, we connect $Y$ to a maximal face $Y'$ of $\Delta_\prec(P_j)$. By Lemma 4(d) and (6),

$$\Delta_\prec(P) \quad = \quad \Delta_\prec(init_\omega(P)) \quad = \quad \Delta_\prec(P_1) \cup \Delta_\prec(P_2) \cup \ldots \cup \Delta_\prec(P_r).$$

Therefore both $X'$ and $Y'$ are maximal faces of $\Delta_\prec(P)$. We connect $X'$ and $Y'$ by a chain of maximal faces in $\Delta_\prec(P)$, using Theorem 1. This gives us a strong connection between $X'$ and $Y'$ in $\Delta_\omega(P)$, by Lemma 6. In summary, we have constructed a strong connection between $X$ and $Y$ in $\Delta_\omega(P)$. $\triangleleft$

The first part of Theorem 2 (and hence of Theorem 1) can be generalized to unmixed ideals. Theorem 2 and Lemma 4(d) have the following corollary.

**Corollary 1** *Let $I$ be an unmixed ideal in $K[x_1,\ldots,x_n]$ and $\omega \in \mathbf{N}^n$ any weight vector. Then the initial complex $\Delta_\omega(I)$ is pure of dimension $dim(I)-1$.*

As an application of this result we obtain the following algorithm for computing the dimension of an unmixed ideal $I$. Compute a Gröbner basis $G$ for $I$ with respect to any term order $\prec$, and read off the set of initial monomials $M := \{init_\prec(g) : g \in G\}$. Initialize $X_0 := \emptyset$. For $i = 1,2,\ldots,n$ set

$$X_i \quad := \quad \begin{cases} X_{i-1} \cup \{x_i\} & \text{if } M \cap K[X_{i-1} \cup \{x_i\}] = \emptyset \\ X_{i-1} & \text{otherwise.} \end{cases}$$

The cardinality of $X_n$ equals the dimension of $I$. This algorithm generalizes [12, Cor. 2.2].

# 4 Examples

We illustrate the construction of initial complexes for three important classes of prime ideals. For experts in algebraic combinatorics we note that each initial complex $\Delta_\prec(I)$ in this section is a shellable ball (hence Cohen-Macaulay) of dimension $dim(I)-1$.

**4.1. Toric Varieties**
Let $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ be a spanning subset of the integer lattice $\mathbf{Z}^d$. The *toric ideal* $I_\mathcal{A}$ associated with $\mathcal{A}$ is defined as the kernel of the $K$-algebra homomorphism

$$K[x_1,\ldots,x_n] \,\rightarrow\, K[t_1,t_1^{-1},\ldots,t_d,t_d^{-1}], \quad x_i \,\mapsto\, t_1^{a_{i1}} t_2^{a_{i2}} \cdots t_d^{a_{id}},$$

where $a_i = (a_{i1}, a_{i2}, \ldots, a_{id})$. This prime ideal defines a $d$-dimensional affine toric variety in $\bar{K}^n$. If there exists a linear functional $h : \mathbf{Z}^d \rightarrow \mathbf{Q}$ with $h(a_1) = \ldots = h(a_n) = 1$, then $I_\mathcal{A}$ is a homogeneous prime ideal, defining a $(d-1)$-dimensional projective variety in $\mathbf{P}^{n-1}$.

The independence complex $\Delta(I_\mathcal{A})$ consists of those subsets of variables $\{x_{i_1},\ldots,x_{i_p}\}$ for which $\{a_{i_1},\ldots,a_{i_p}\}$ is linearly independent in $\mathbf{Q}^d$. Thus $\Delta(I_\mathcal{A})$ is the matroid complex of the linear rank $d$ matroid of the vector configuration $\mathcal{A}$. If $I_\mathcal{A}$ is homogeneous, then this is the matroid associated with the $(d-1)$-dimensional polytope $Q = conv(\mathcal{A})$.

The initial complexes $\Delta_\prec(I_\mathcal{A})$, where $\prec$ is any term order, are precisely the *regular triangulations* of $\mathcal{A}$. This result was proved for the homogeneous case in [18]. It can be extended to all toric ideals using the methods in [1]. If $I_\mathcal{A}$ is homogeneous, then the regular triangulations of $\mathcal{A}$ are the regular triangulations of the polytope $Q$ having vertices in $\mathcal{A}$.

For general $\omega \in \mathbf{N}^n$, the initial complex $\Delta_\omega(I_\mathcal{A})$ corresponds to the regular polyhedral subdivision of $\mathcal{A}$ defined by $\omega$. More precisely, $\Delta_\omega(I_\mathcal{A})$ is the union of the matroid complexes $\Delta(I_{\mathcal{A}'})$, where $\mathcal{A}'$ runs over all cells in this subdivision. This follows from Lemma 1(d). In fact, using toric ideals, all our results can be interpreted in terms of polyhedral geometry.

### 4.2. Algebras with straightening law

We refer to the axiomatic theory of *Hodge algebras* or *algebras with straightening law* due to DeConcini, Eisenbud and Procesi [5],[6]. Let $\mathcal{P}$ be a poset on $\{x_1, \ldots, x_n\}$, and let $A$ be a graded $K$-algebra with straightening law over $\mathcal{P}$. We write $A = K[x_1, \ldots, x_n]/I$, where $I$ is the homogeneous ideal generated by the straightening relations.

Consider any variable ordering which is a linear extension of $\mathcal{P}$, and let $\prec$ be the induced degree reverse lexicographic term order on $K[x_1, \ldots, x_n]$. It is known (see e.g. [9]) that the quadratic straightening relations are a Gröbner basis for $I$ with respect to $\prec$. This implies that the initial complex $\Delta_\prec(I)$ equals the *chain complex* of the poset $\mathcal{P}$. A subset $X$ of the variables is a face of $\Delta_\prec(I)$ if and only if $X$ is a chain in $\mathcal{P}$.

The prototypical example in this theory is the ideal $I_{n,d}$ of the Grassmann variety of $d$-flats in $\bar{K}^n$, given in its Plücker embedding. Here the set of variables equals

$$\Lambda(n, d) \quad = \quad \{\, [i_1 i_2 \ldots i_d] \;:\; 1 \leq i_1 < i_2 < \ldots < i_d \leq n \,\}.$$

Following [19], we fix the lexicographic ordering on the *brackets* $[i_1 i_2 \ldots i_d]$. This ordering is a linear extension of *Young's poset* on $\Lambda(n, d)$, which is defined by

$$[i_1 i_2 \ldots i_d] \leq [j_1 j_2 \ldots j_d] \quad :\Longleftrightarrow \quad (i_1 \leq j_1) \text{ and } (i_2 \leq j_2) \text{ and } \ldots \text{ and } (i_d \leq j_d).$$

The initial complex $\Delta_\prec(I_{n,d})$ of the Plücker ideal is the chain complex of Young's poset.

### 4.3. Determinantal Ideals

Let $K[x_{ij}]$ be the ring of polynomial functions on a generic $m \times n$-matrix, and let $I_r$ denote the ideal generated by the $r \times r$-minors of $(x_{ij})$. The set of variables is partially ordered by setting $x_{ij} \leq x_{kl}$ whenever $i \leq k$ and $j \leq l$. This poset equals $[m] \times [n]$, the product of an $m$-chain and an $n$-chain. Consider any variable ordering which is a linear extension of $[m] \times [n]$, and let $\prec$ be the induced degree reverse lexicographic term order on $K[x_{ij}]$.

It is known [17] that the set of $r \times r$-minors is the reduced Gröbner basis for $I_r$ with respect to $\prec$. This implies that the initial complex $\Delta_\prec(I_r)$ coincides with the *$r$-th order complex* of $[m] \times [n]$. A set $X \subset \{x_{11}, x_{12}, \ldots, x_{mn}\}$ is a face of $\Delta_\prec(I_r)$ if and only if $X$ does not contain an antichain of cardinality $r$ in $[m] \times [n]$. This construction has been extended by Herzog and Trung [11] to more general determinantal and Pfaffian ideals.

For the case of maximal minors ($r = m \leq n$) the initial complex $\Delta_<(I_m)$ with respect to an arbitrary term order $<$ was described in [20]. Identifying $<$ with a coherent matching field $\Lambda$, the maximal faces of $\Delta_<(I_m)$ are precisely the sets $\Omega_\rho$ in (5.1) of [20]. We propose that the combinatorial analysis in [20] be extended to the general case $2 < r < m \leq n$.

**4.4. Triangulations of the triangular prism**

Here is a specific ideal which lies in the intersection of the three families given in 4.1, 4.2 and 4.3. Let $m = 2, n = 3$ and consider the determinantal ideal

$$I_2 \quad = \quad \langle\, x_{11}x_{22} \,-\, x_{12}x_{21}\,,\;\; x_{11}x_{23} \,-\, x_{13}x_{21}\,,\;\; x_{12}x_{23} \,-\, x_{13}x_{22}\,\rangle.$$

The quotient algebra $K[x_{ij}]/I_2$ is an algebra with straightening law over the poset $\mathcal{P} = [2] \times [3]$. We fix the variable ordering $x_{11} < x_{21} < x_{12} < x_{22} < x_{13} < x_{23}$. Let $\prec$ denote the induced reverse lexicographic term order. The initial ideal equals

$$init_{\prec}(I_2) \quad = \quad \langle\, x_{12}x_{21}\,,\; x_{13}x_{21}\,,\; x_{13}x_{22}\,\rangle.$$

The initial complex $\Delta_{\prec}(I_2)$ is the chain complex of the poset $[2]\times[3]$. It has three maximal faces $\{x_{11}, x_{21}, x_{22}, x_{23}\}$, $\{x_{11}, x_{12}, x_{22}, x_{23}\}$ and $\{x_{11}, x_{12}, x_{13}, x_{23}\}$. Geometrically, $\Delta_{\prec}(I_2)$ is a regular triangulation of the vector configuration $\mathcal{A} = \{(0,0,0,1),(1,0,0,1),(0,1,0,1),(0,0,1,1),(1,0,1,1),(0,1,1,1)\}$. Indeed, the homogeneous toric ideal $I_{\mathcal{A}}$ coincides with the determinantal ideal $I_2$. The polytope $Q = conv(\mathcal{A})$ has dimension 3; it is a triangular prism. Thus $\Delta_{\prec}(I_2)$ is a triangulation of the triangular prism into three tetrahedra. There are six triangulations, one for each of the six distinct initial monomial ideals of $I_2 = I_{\mathcal{A}}$.

**4.5. Suggestion for further research**

Our main result (Theorem 1) gives a necessary condition for a simplicial complex $\Delta$ to be the initial complex $\Delta_{\prec}(P)$ of a prime ideal $P$. We consider it a challenging open problem to find a criterion, which is both necessary and sufficient.

# References


[1] L.J. Billera, I.M. Gel'fand and B. Sturmfels: Duality and minors of secondary polyhedra, *Journal of Combinatorial Theory* **B**, to appear.

[2] A. Björner: Homology and shellability of matroids and geometric lattices, in *Matroid Applications* (ed. Neil White), pp. 226–283, Cambridge University Press, 1992.

[3] B. Buchberger: Gröbner bases: An algorithmic method in polynomial ideal theory, in *Multidimensional Systems Theory* (ed. N.K. Bose), pp. 184–232, D. Reidel Publishing Company, Dortrecht-Boston-Lancaster, 1985.

[4] G. Carrà-Ferro: Some properties of lattice points and their applications to differential algebra, *Communications in Algebra*, 15:2625–2632, 1987.

[5] C. De Concini, D. Eisenbud and C. Procesi: Hodge algebras, *Astérique 91*, 1982.

[6] D. Eisenbud: Introduction to algebras with straightening laws, in *Ring Theory and Algebra* (ed. B.R. McDonald), Proceedings of the third Oklahoma conference, Marcel Dekker, New York, 1980.

[7] W. Fulton and J. Hansen: A connectedness theorem for projective varieties, with applications to intersections and singularities of mappings, *Annals of Mathematics*, 110:159–166, 1979.

[8] W. Fulton and R. Lazarsfeld: Connectivity and its applications in algebraic geometry, in *Algebraic Geometry* (eds. A. Libgober and P. Wegreich), Springer Lecture Notes in Mathematics 862, pp. 26–92, 1981.

[9] H.-G. Gräbe: Über Streckungsringe, *Beiträge zur Geometrie und Algebra*, 23:85–100, 1986.

[10] W. Gröbner: *Algebraische Geometrie II*, Bibliographisches Institut Mannheim, 1970.

[11] J. Herzog and N.V. Trung: Gröbner bases and multiplicity of determinantal and Pfaffian ideals, *Advances in Mathematics*, to appear.

[12] H. Kredel and V. Weispfenning: Computing dimension and independent sets for polynomial ideals, *Journal of Symbolic Computation*, 6:213–248, 1988.

[13] T. Mora and L. Robbiano: The Gröbner fan of an ideal, *Journal of Symbolic Computation*, 6:183–208, 1988.

[14] I.R. Shafarevich: *Basic Algebraic Geometry*, Springer, 1977.

[15] R. Stanley: *Combinatorics and Commutative Algebra*, Birkhäuser, Boston, 1983.

[16] R. Stanley: Cohen Macaulay complexes, in *Higher Combinatorics* (ed. M. Aigner), pp. 51–62, NATO Advanced Study Institute Series, D. Reidel Publishing Company, Dortrecht-Boston, 1977.

[17] B. Sturmfels: Gröbner bases and Stanley decompositions of determinantal rings, *Mathematische Zeitschrift*, 205:137–144, 1990.

[18] B. Sturmfels: Gröbner bases of toric varieties, *Tôhoku Math. J.*, 43:249–261, 1991.

[19] B. Sturmfels and N. White: Gröbner bases and invariant theory, *Advances in Mathematics*, 76:245–259, 1989.

[20] B. Sturmfels and A. Zelevinsky: Maximal minors and their leading terms, *Advances in Mathematics*, to appear.