Asymptotic behaviour of multivariate default probabilities and default correlations under stress

N. Packham, M. Kalkbrener, L. Overbeck*

January 14, 2015

Abstract

We investigate default probabilities and default correlations of Merton-type credit portfolio models in stress scenarios where a common risk factor is truncated. For elliptically distributed asset variables, the asymptotic limits of default probabilities and default correlations depend on the max-domain of attraction of the asset variables. In the regularly varying case, we derive an integral representation for multivariate default probabilities, which turn out to be strictly smaller than 1. Default correlations are in $(0, 1)$. In the rapidly varying case, asymptotic multivariate default probabilities are 1 and asymptotic default correlations are 0.

Keywords: financial risk management, credit portfolio modelling, stress testing, elliptic distribution, max-domain of attraction

MSC classification: 60G70, 91G40

1 Introduction

We analyse the behaviour of structural credit portfolio models under stress depending on the joint distribution of the stochastic variables of the model. To cover a wide range of light-tailed to heavy-tailed distributions we consider the family of elliptical distributions. More formally, let $\boldsymbol{Z} = (Z_0, \dots, Z_d)^T$ be a random vector following an elliptical distribution. In our setting, Z_0 will be interpreted as a risk factor of the credit portfolio model and $Z_1, \dots, Z_d$ as asset return variables of d firms. The default of the i-th firm is represented by $\{Z_i \leq D_i\}$ for a given default threshold $D_i \in \mathbb{R}$ and the corresponding default probability (PD) is $\mathbf{P}(Z_i \leq D_i) = \mathbb{E}(\mathbf{1}_{\{Z_i \leq D_i\}})$. Default correlations are defined as the correlations of $\mathbf{1}_{\{Z_i \leq D_i\}}$ and $\mathbf{1}_{\{Z_j \leq D_j\}}$. Throughout default probabilities $\mathbf{P}(Z_i \leq D_i)$, $i = 1, \dots, d$, and correlations of $Z_0, Z_1, \dots, Z_d$ are assumed to be in $(0, 1)$.

The objective of the paper is to analyse the impact of stress on multivariate default probabilities and on default correlations. Stress scenarios are specified by truncating the risk factor Z_0, i.e., by conditioning on $\{Z_0 \leq C\}$, where $C \in \mathbb{R}$ is the stress level. The limits of conditional default probabilities and default correlations as $C \to -\infty$ depend on whether the random variables $Z_0, Z_1, \dots, Z_d$ are in the max-domain of attraction of the Fréchet or the Gumbel distribution, or more generally, on whether the tail distribution functions $\mathbf{P}(Z_i > \cdot)$, $i = 0, 1, \dots, d$,

*Natalie Packham, Department of Finance, Frankfurt School of Finance & Management, Sonnemannstr. 9–11, 60314 Frankfurt am Main, Germany. Phone: +49 69 154008-723, Fax: +49 69 154008-4723, Email: n.packham@fs.de.

Michael Kalkbrener, Deutsche Bank AG, Taunusanlage 12, 60325 Frankfurt am Main, Germany.

Ludger Overbeck, Institute of Mathematics, University of Giessen, Arndtstrasse 2, 35392 Giessen, Germany.

The views expressed in this paper are those of the authors and do not necessarily reflect the position of Deutsche Bank AG.

are regularly varying or rapidly varying. In the rapidly varying case, we show that default probabilities $\mathbf{P}(Z_1 \leq D_1, \ldots, Z_d \leq D_d | Z_0 \leq C)$, $d \geq 1$, converge to 1 under extreme stress and default correlations converge to 0. In models with regularly varying tail distribution functions, we derive an integral representation and establish that the limits of conditional default probabilities and default correlations attain values in $(0, 1)$. The special case when $d = 1$ has been studied by (Abdous *et al.*, 2005; Balakrishnan and Hashorva, 2011).

With the integral representation for conditional multivariate default probabilities one can calculate tail risk measures such as value-at-risk and expected shortfall in the limit $C \to -\infty$. Kalkbrener and Packham (2015b)use this technique to compare capital reserves and capital requirements in light- and heavy-tailed models.

It is interesting to note that asymptotic default correlations behave like tail dependence, which is positive for heavy-tailed random variables, and converges to 0 as the tail index tends to infinity, that is, to the light-tailed case, see (Schmidt, 2002; Klüppelberg *et al.*, 2008; Hult and Lindskog, 2002). Asymptotic default probabilities, on the other hand, display the opposite behaviour, i.e., they increase with the tail index. We illustrate the fundamental difference between asymptotic stressed PD's and tail dependence by considering the tail dependence function $\lim_{C \to -\infty} \mathbf{P}(Z_i \leq x\, C | Z_0 \leq C)$, $x \in \mathbb{R}$, which specializes to asymptotic stressed PD's as well as tail dependence at $x = 0$ and $x = 1$ respectively.

Related literature on the existence of limiting distributions of a random vector conditional on one component being large is e.g. Berman (1983); Heffernan and Resnick (2007); Hashorva (2006, 2009); Fougeres and Soulier (2010).

2 Credit portfolio modelling

The progenitor of all structural credit portfolio models is the model of Merton (Merton, 1974), which links the default of a firm to the relationship between its assets and the liabilities at the end of a given time period $[0, T]$. More precisely, the i-th counterparty defaults if its asset return (ability-to-pay variable) Z_i falls below a default threshold $D_i \in \mathbb{R}$; the default event at time T is $\{Z_i \leq D_i\}$. The portfolio loss variable is

$$L := \sum_{i=1}^{d} l_i \, \mathbf{1}_{\{Z_i \leq D_i\}}, \tag{1}$$

with d the number of counterparties and l_i the loss-at-default of the i-th counterparty. To reflect risk concentrations, each Z_i is decomposed into a sum of systematic factors $X_1, \ldots, X_m$, such as geographic regions or industries, and a firm-specific factor ε_i, that is,

$$Z_i = \sqrt{R_i^2} \sum_{j=1}^{m} w_{ij} X_j + \sqrt{1 - R_i^2} \varepsilon_i, \quad i = 1, \ldots, d. \tag{2}$$

The impact of the risk factors on Z_i is determined by $R_i^2 \in [0, 1]$ and the factor weights $w_{ij} \in \mathbb{R}$.

To quantify portfolio risk, measures of risk are applied to the portfolio loss distribution (1). The expected loss $\mathbb{E}(L) = \sum_{i=1}^{d} l_i \, \mathbf{P}(Z_i \leq D_i)$ is used for specifying credit reserves. Capital requirements for covering unexpected losses are typically derived from the value-at-risk $\mathrm{VaR}_\beta(L)$ for a predefined probability $\beta \in (0, 1)$, where $\mathrm{VaR}_\beta(L)$ is simply defined as the β-quantile $\inf\{x \in \mathbb{R} : \mathbf{P}(L \leq x) \geq \beta\}$ of L. Obviously, the default probabilities and risk concentrations specified by the dependence structure of the default variables $\mathbf{1}_{\{Z_i \leq D_i\}}$ determine the value-at-risk of the credit portfolio. Default correlations

$$\mathrm{Corr}(\mathbf{1}_{\{Z_i \leq D_i\}}, \mathbf{1}_{\{Z_j \leq D_j\}}) = \frac{\mathbf{P}(Z_i \leq D_i, Z_j \leq D_j) - p_i \, p_j}{\sqrt{p_i(1 - p_i) p_j (1 - p_j)}},$$

where $p_i := \mathbf{P}(Z_i \le D_i)$, $i = 1, \ldots, d$, are used as a measure of dependence by portfolio management to identify risk concentrations on counterparty level.

In a stress test, a credit portfolio is evaluated under the assumption of adverse economic conditions. A natural way for implementing stress tests in portfolio models is to translate the stress scenario into constraints on risk factors. In a common setup, the constraints are formalised by truncating a risk factor variable Z_0 (which is typically one of the X_i), that is, by setting $Z_0 \le C$, for a stress level $C \in \mathbb{R}$, e.g. (Bonti *et al.*, 2006; Duellmann and Erdelmeier, 2009; Kalkbrener and Packham, 2015a). Portfolio risk is evaluated under the conditional distribution $\mathbf{P}(\cdot|Z_0 \le C)$.

The standard approach in credit risk management is to model the risk factors and ability-to-pay variables through a multivariate Gaussian distribution. Since the purpose of this paper is to analyse the impact of stress scenarios under different distribution assumptions, we use a more general framework and consider elliptical distributions instead, as they cover a variety of light-tailed to heavy-tailed distributions and include distributions commonly used in financial modelling.

3 Multivariate default probabilities and default correlations under stress

Let the random vector $\boldsymbol{Z} = (Z_0, \ldots, Z_d)^T$ follow an elliptical distribution with representation $\boldsymbol{Z} \stackrel{\mathcal{L}}{=} GA\boldsymbol{U}$, where $G > 0$ is a scalar random variable, the so-called *mixing variable*, A is a deterministic $(d+1) \times (d+1)$ matrix with $AA^T := \Sigma$, which in turn is a $(d+1) \times (d+1)$ nonnegative definite symmetric matrix of rank $d+1$, and $\boldsymbol{U}$ is a $(d+1)$-dimensional random vector uniformly distributed on the unit sphere $\mathcal{S}_{d+1} := \{\boldsymbol{z} \in \mathbb{R}^{d+1} : \boldsymbol{z}^T\boldsymbol{z} = 1\}$, and $\boldsymbol{U}$ is independent of G.

We assume that $\boldsymbol{Z}$ is standardised so that $\Sigma = AA^T$ is the correlation matrix of $\boldsymbol{Z} = (Z_0, \ldots, Z_d)^T$. The correlation of Z_i and Z_j is denoted by ρ_{ij}, $i, j = 0, 1, \ldots, d$. We further assume that these correlations are positive, i.e., $\rho_{ij} > 0$. Cases with zero or negative correlations can be treated analogously. Denote by $A_{i\cdot}$ the i-th row of A and let $F_{\boldsymbol{U}}$ denote the uniform distribution on $\mathcal{S}_{d+1}$.

For a positive, Lebesgue-measurable function h on $(0, \infty)$, we write $h \in RV_\alpha$ if h is regularly varying with index $\alpha \in \mathbb{R}$, and $h \in RV_{-\infty}$ if h is rapidly varying with index $-\infty$. For details on regularly varying and rapidly varying functions see Bingham *et al.* (1987). It is well-known that a random variable X is in the max-domain of attraction of the Fréchet-distribution with index $\alpha > 0$ if and only if its tail function $\mathbf{P}(X > \cdot) \in RV_{-\alpha}$, see e.g. Embrechts *et al.* (1997). Similarly, if X is in the Gumbel max-domain of attraction, then $\mathbf{P}(X > \cdot) \in RV_{-\infty}$. In the special case of elliptically distributed random vectors, $\mathbf{P}(G > \cdot) \in RV_{-\alpha}$ implies $\mathbf{P}(Z_i > \cdot) \in RV_{-\alpha}$, $i = 0, \ldots, d$, see Breiman (1965), resp. Theorem 7.35 of McNeil *et al.* (2005). Berman Berman (1983) shows that if G is in the Gumbel max-domain of attraction, then Z_i is also in the Gumbel max-domain of attraction. Conversely, the following proposition shows that the variation of $\mathbf{P}(Z_i > \cdot)$ determines the variation of $\mathbf{P}(G > \cdot)$.

Proposition 1. *If $\mathbf{P}(Z_i > \cdot) \in RV_{-\alpha}$, $\alpha > 0$, for some $i \in \{0, \ldots, d\}$, then $\mathbf{P}(G > \cdot) \in RV_{-\alpha}$, and likewise for $RV_{-\infty}$.*

Proof. For $RV_{-\alpha}$ the statement follows from Theorem 4.2 of Jacobsen *et al.* (2009). For $RV_{-\infty}$, we write $\boldsymbol{U}$ in polar coordinates and assume w.l.o.g. that $Z_i = G\cos\theta$, where θ is uniformly distributed in $[-\pi,\pi]$. Consider first the case $t>1$:

$$0=\lim_{x\to\infty}\frac{\mathbf{P}(Z_i>tx)}{\mathbf{P}(Z_i>x)}=\lim_{x\to\infty}\frac{\mathbf{P}(|Z_i|>tx)}{\mathbf{P}(|Z_i|>x)}=\lim_{x\to\infty}\frac{\mathbf{P}(G>tx/|\cos\theta|)}{\mathbf{P}(G>x)}\,\frac{\mathbf{P}(G>x)}{\mathbf{P}(G>x/|\cos\theta|)}$$
$$=\lim_{x\to\infty}\int_{-\pi}^{\pi}\frac{\mathbf{P}(G>tx/|\cos\tilde\theta|)}{\mathbf{P}(G>x)}\,\frac{\mathbf{P}(G>x)}{\mathbf{P}(G>x/|\cos\theta|)}\,\mathrm{d}\tilde\theta,$$

where the last step follows from the independence of θ and G. Since the second term is greater than one, the first term must converge to zero. Hence, it follows that $\lim_{x\to\infty}\mathbf{P}(G>tx)/\mathbf{P}(G>x)=0$ for any $t>1$. For the case $t<1$, we write $\mathbf{P}(G>tx)/\mathbf{P}(G>x)$ in the form $1/(\mathbf{P}(G>(1/t)tx)/\mathbf{P}(G>tx))$. Since $1/t>1$ we obtain $\lim_{x\to\infty}\mathbf{P}(G>(1/t)tx)/\mathbf{P}(G>tx)=0$ and therefore $\lim_{x\to\infty}\mathbf{P}(G>tx)/\mathbf{P}(G>x)=\infty$. □

3.1 Integral representation for multivariate default probabilities

The following theorem treats multivariate asymptotic default probabilities of arbitrary dimension.

Theorem 2. *(i) If $\mathbf{P}(G>\cdot)\in RV_{-\alpha}$, then*

$$\lim_{C\to-\infty}\mathbf{P}(Z_1\le D_1,\dots,Z_d\le D_d|Z_0\le C)$$
$$=\int_{\boldsymbol{u}\in\mathcal{S}_{d+1},A_{0\cdot}\boldsymbol{u}>0,\dots,A_{d\cdot}\boldsymbol{u}>0}(A_{0\cdot}\boldsymbol{u})^\alpha\,\mathrm{d}F_{\boldsymbol{U}}(\boldsymbol{u})\left(\int_{\boldsymbol{u}\in\mathcal{S}_{d+1},A_{0\cdot}\boldsymbol{u}>0}(A_{0\cdot}\boldsymbol{u})^\alpha\,\mathrm{d}F_{\boldsymbol{U}}(\boldsymbol{u})\right)^{-1}.$$

(ii) If $\mathbf{P}(G>\cdot)\in RV_{-\infty}$, then $\lim_{C\to-\infty}\mathbf{P}(Z_1\le D_1,\dots,Z_d\le D_d|Z_0\le C)=1$.

Note that in the regularly varying case, default probabilities are strictly smaller than 1, provided the variables are not perfectly correlated.

The following Lemma is used in the proof of the rapidly varying case, (ii). Under the slightly more restricted condition that the mixing variable G is in the Gumbel max-domain of attraction, the statement follows from Theorem 2 (ii) of Abdous *et al.* (2005), see also Theorem 4.1 in Berman (1983).

Lemma 3. *If $\mathbf{P}(G>\cdot)\in RV_{-\infty}$, then $\lim_{C\to-\infty}\mathbf{P}(Z_1\le 0|Z_0\le C)=1$.*

Proof. We have

$$\mathbf{P}(Z_1\le 0|Z_0\le C)=\mathbf{P}(Z_1>0|Z_0>C)=\frac{\mathbf{P}(Z_0>C,Z_1>0))}{\mathbf{P}(Z_0>C)}$$
$$=\mathbf{P}\left(G>\frac{C}{A_{0\cdot}\boldsymbol{U}},A_{0\cdot}\boldsymbol{U}>0\right)\left(\mathbf{P}\left(G>\frac{C}{A_{0\cdot}\boldsymbol{U}}\right)\right)^{-1}$$
$$=\int_{\boldsymbol{u}\in\mathcal{S}_2,A_{i\cdot}\boldsymbol{u}>0,i=0,1}\mathbf{P}\left(G>\frac{C}{A_{0\cdot}\boldsymbol{u}}\right)F_{\boldsymbol{U}}(\mathrm{d}\boldsymbol{u})\left(\int_{\boldsymbol{u}\in\mathcal{S}_2,A_{0\cdot}\boldsymbol{u}>0}\mathbf{P}\left(G>\frac{C}{A_{0\cdot}\boldsymbol{u}}\right)F_{\boldsymbol{U}}(\mathrm{d}\boldsymbol{u})\right)^{-1}.$$

Write $\boldsymbol{u}\in\mathcal{S}_2$ in polar coordinates as $\boldsymbol{u}=(\cos\theta,\sin\theta)$, $\theta\in[-\pi,\pi]$, and let A be the Cholesky decomposition of the correlation matrix, i.e., $A_{0\cdot}=(1,0)^T$, $A_{1\cdot}=(\rho,\sqrt{1-\rho^2})$ with $\rho:=\rho_{01}$ the correlation of Z_0,Z_1. Hence, $A_{0\cdot}\boldsymbol{u}=\cos\theta>0$ if $\theta\in(-\pi/2,\pi/2)$ and $A_{1\cdot}\boldsymbol{u}=\rho\cos\theta+\sqrt{1-\rho^2}\sin\theta=\sin(\theta+\arcsin\rho)>0$ if $\theta\in(-\arcsin\rho,\pi-\arcsin\rho)$. It follows that

$$\int_{\boldsymbol{u}\in\mathcal{S}_2,A_{i\cdot}\boldsymbol{u}>0,i=0,1}\mathbf{P}\left(G>\frac{C}{A_{0\cdot}\boldsymbol{u}}\right)F_{\boldsymbol{U}}(\mathrm{d}\boldsymbol{u})=\int_{-\arcsin\rho}^{\pi/2}\mathbf{P}\left(G>\frac{C}{\cos\theta}\right)\mathrm{d}\theta$$
$$=\int_{-\pi/2}^{\pi/2}\mathbf{P}\left(G>\frac{C}{\cos\theta}\right)\mathrm{d}\theta-\int_{\arcsin\rho}^{\pi/2}\mathbf{P}\left(G>\frac{C}{\cos\theta}\right)\mathrm{d}\theta$$

and

$$\int_{\boldsymbol{u}\in\mathcal{S}_2, A_0.\boldsymbol{u}>0} \mathbf{P}\left(G > \frac{C}{A_0.\boldsymbol{u}}\right) F_{\boldsymbol{U}}(\mathrm{d}\boldsymbol{u}) = \int_{-\pi/2}^{\pi/2} \mathbf{P}\left(G > \frac{C}{\cos\theta}\right) \mathrm{d}\theta.$$

Since $\rho > 0$, we have $\cos\theta < \cos(\arcsin\rho) = \sqrt{1-\rho^2}$ for $\theta \in (\arcsin\rho, \pi/2)$. Hence, by definition of rapidly varying functions and by Dominated Convergence,

$$\lim_{C\to\infty} \frac{\mathbf{P}\left(G > \frac{C}{\sqrt{1-\rho^2}} \frac{\sqrt{1-\rho^2}}{\cos\theta}\right)}{\mathbf{P}\left(G > \frac{C}{\sqrt{1-\rho^2}}\right)} = \begin{cases} 0, & \text{if } \theta \in (\arcsin\rho, \pi/2), \\ \infty, & \text{if } \theta \in (-\arcsin\rho, \arcsin\rho), \end{cases}$$

and the claim follows by putting everything together. □

Proof of Theorem 2. We first give a proof for the special case $D_i = 0$ for $i = 1, \dots, d$.

The assumption $G > 0$ implies that $(Z_0, \dots, Z_d)$ is symmetric with marginals that have continuous distribution functions, and hence we can write

$$\lim_{C\to-\infty} \mathbf{P}(Z_1 \le 0, \dots, Z_d \le 0 | Z_0 \le C) = \lim_{C\to\infty} \frac{\mathbf{P}(Z_0 > C, Z_1 > 0, \dots, Z_d > 0)}{\mathbf{P}(Z_0 > C)}. \tag{3}$$

For the numerator

$$\begin{aligned}\mathbf{P}(Z_0 > C, Z_1 > 0, \dots, Z_d > 0) &= \mathbf{P}\left(G > \frac{C}{A_0.\boldsymbol{U}}, A_0.\boldsymbol{U} > 0, \dots, A_d.\boldsymbol{U} > 0\right) \\ &= \int_{\boldsymbol{u}\in\mathcal{S}_{d+1}, A_i.\boldsymbol{u}>0, i=0,\dots,d} \mathbf{P}\left(G > \frac{C}{A_0.\boldsymbol{u}}\right) F_{\boldsymbol{U}}(\mathrm{d}\boldsymbol{u}). \end{aligned} \tag{4}$$

For *(i)*, it follows from $\mathbf{P}(G > \cdot) \in RV_{-\alpha}$ that

$$\lim_{C\to\infty} \frac{\mathbf{P}\left(G > C/(A_0.\boldsymbol{u})\right)}{\mathbf{P}(G > C)} = (A_0.\boldsymbol{u})^\alpha, \quad \text{for } A_0.\boldsymbol{u} > 0.$$

Applying Potter's bounds (de Haan and Ferreira, 2006, Proposition B.1.9) and since the right-hand side is integrable, we obtain by Dominated Convergence that

$$\lim_{C\to\infty} \int_{\boldsymbol{u}\in\mathcal{S}_{d+1}, A_i.\boldsymbol{u}>0, i=0,\dots,d} \frac{\mathbf{P}\left(G > C/(A_0.\boldsymbol{u})\right)}{\mathbf{P}(G > C)} F_{\boldsymbol{U}}(\mathrm{d}\boldsymbol{u}) = \int_{\boldsymbol{u}\in\mathcal{S}_{d+1}, A_i.\boldsymbol{u}>0, i=0,\dots,d} (A_0.\boldsymbol{u})^\alpha F_{\boldsymbol{U}}(\mathrm{d}\boldsymbol{u}).$$

Applying the same method to the denominator of Equation (3) completes the proof of *(i)* for the case $D_i = 0$.

For *(ii)*, it suffices to consider the case $d = 1$, i.e., $\lim_{C\to\infty} \mathbf{P}(Z_1 > 0 | Z_0 > C) = 1$, which is established in Lemma 3, since the general case follows from $\mathbf{P}(Z_1 > 0, \dots, Z_d > 0 | Z_0 > C) \ge 1 - \sum_{i=1}^d (1 - \mathbf{P}(Z_i > 0 | Z_0 > C))$.

It remains to show that

$$\lim_{C\to-\infty} \mathbf{P}(Z_1 \le 0, \dots, Z_d \le 0 | Z_0 \le C) = \lim_{C\to-\infty} \mathbf{P}(Z_1 \le D_1, \dots, Z_d \le D_d | Z_0 \le C), \tag{5}$$

for arbitrary $D_1, \dots, D_d$. Let $i \in \{1, \dots, d\}$ and $a > 0$. Note that for $C < -|D_i|/a$,

$$\begin{aligned}\mathbf{P}(Z_i - aZ_0 \le 0 | Z_0 \le C) \le \mathbf{P}(Z_i \le aC | Z_0 \le C) &< \mathbf{P}(Z_i \le D_i | Z_0 \le C) \\ &< \mathbf{P}(Z_i \le -aC | Z_0 \le C) \le \mathbf{P}(Z_i + aZ_0 \le 0 | Z_0 \le C).\end{aligned}$$

Hence,

$$\lim_{C\to-\infty} \mathbf{P}(Z_1 - aZ_0 \le 0, Z_2 \le 0, \ldots, Z_d \le 0 | Z_0 \le C)$$
$$\le \lim_{C\to-\infty} \mathbf{P}(Z_1 \le D_1, Z_2 \le 0, \ldots, Z_d \le 0 | Z_0 \le C)$$
$$\le \lim_{C\to-\infty} \mathbf{P}(Z_1 + aZ_0 \le 0, Z_2 \le 0, \ldots, Z_d \le 0 | Z_0 \le C).$$

Since Equation (4) is continuous in A_1., it follows that $\lim_{C\to-\infty} \mathbf{P}(Z_1 + aZ_0 \le 0, Z_2 \le 0, \ldots, Z_d \le 0 | Z_0 \le C)$ is a continuous function in $a \in \mathbb{R}$, hence

$$\lim_{C\to-\infty} \mathbf{P}(Z_1 \le D_1, Z_2 \le 0, \ldots, Z_d \le 0 | Z_0 \le C) = \lim_{C\to-\infty} \mathbf{P}(Z_1 \le 0, Z_2 \le 0, \ldots, Z_d \le 0 | Z_0 \le C).$$

and therefore (5) is obtained by reiterating this argument. □

It is interesting to note that although default thresholds $D_1, \ldots, D_d$ determine the unconditional default probabilities $\mathbf{P}(Z_i \le D_i)$, $\lim_{C\to-\infty} \mathbf{P}(Z_1 \le D_1, \ldots, Z_d \le D_d | Z_0 \le C)$ does not depend on the D_i. In the limit, stressed default probabilities depend only on the dependence structure of the Z_i.

3.2 Analytic formulas for univariate and bivariate default probabilities

Even without further specification, the limiting distribution of Theorem 2, part (i), can be efficiently determined using Monte Carlo simulation. In this section, we express special cases of the integral in Theorem 2(i) in terms of beta functions and Student t-distribution functions. The first part of Theorem 4 covers the univariate case $d = 1$, which corresponds to stressed default probabilities, whereas the second part deals with stressed bivariate default probabilities. The univariate result follows easily from the integral representation in Theorem 2(i). Despite being already established in Theorem 2(i) of Abdous *et al.* (2005), we provide the short proof as it demonstrates the technique to obtain concrete expressions for $d \ge 1$.

We assume that $\rho_{12} \ge \rho_{01}\rho_{02}$, which expresses that the specific components of Z_1 and Z_2 are correlated in a non-negative way. The sole reason for this assumption is to avoid awkward case differentiations, and it can easily be lifted.

Let t_ν denote the Student-t distribution function with parameter ν, and let $B(a, b)$ denote the Beta function with parameters a and b.

Theorem 4. *Let* $\mathbf{P}(G > \cdot) \in RV_{-\alpha}$.

(i) For $d = 1$,

$$\lim_{C\to-\infty} \mathbf{P}(Z_1 \le D_1 | Z_0 \le C) = t_{\alpha+1}\left(\frac{\sqrt{\alpha+1}\,\rho_{01}}{\sqrt{1-\rho_{01}^2}}\right) \in [1/2, 1).$$

(ii) For $d = 2$,

$$\lim_{C\to-\infty} \mathbf{P}(Z_1 \le D_1, Z_2 \le D_2 | Z_0 \le C) = \frac{1}{2} t_{\alpha+1}\left(\frac{\sqrt{(\alpha+1)}\,t}{\sqrt{1-t^2}}\right)$$
$$+ \int_{-\arcsin t}^{\pi/2} \left[\frac{1}{2} - t_{\alpha+2}\left(-\left|\frac{\sqrt{\alpha+2}}{q_3(\varphi)}\right|\right)\right] (\cos\varphi)^\alpha \,\mathrm{d}\varphi \left(B\left(\frac{1}{2}, \frac{\alpha+1}{2}\right)\right)^{-1}$$
$$+ \int_{-\arcsin \rho_{01}}^{-\arcsin t} t_{\alpha+2}\left(-\left|\frac{\sqrt{\alpha+2}}{q_3(\varphi)}\right|\right) (\cos\varphi)^\alpha \,\mathrm{d}\varphi \left(B\left(\frac{1}{2}, \frac{\alpha+1}{2}\right)\right)^{-1},$$

where $q_3(\varphi) = \dfrac{\sqrt{1-\rho_{02}^2-q_1^2}}{\rho_{02}\cos\varphi + q_1\sin\varphi}$ *and* $q_1 = \dfrac{\rho_{12}-\rho_{01}\rho_{02}}{\sqrt{1-\rho_{01}^2}}$ *and* $t := \rho_{01} \wedge \frac{\rho_{02}}{\sqrt{q_1^2+\rho_{02}^2}}$.

Proof. *(i)* By Theorem 2(i),

$$\lim_{C\to-\infty}\mathbf{P}(Z_1\le D_1|Z_0\le C)=\int_{\boldsymbol{u}\in\mathcal{S}_2,A_{0\cdot}\boldsymbol{u}>0,A_{1\cdot}\boldsymbol{u}>0}(A_{0\cdot}\boldsymbol{u})^{\alpha}\,\mathrm{d}F_{\boldsymbol{U}}(\boldsymbol{u})\left(\int_{\boldsymbol{u}\in\mathcal{S}_2,A_{0\cdot}\boldsymbol{u}>0}(A_{0\cdot}\boldsymbol{u})^{\alpha}\,\mathrm{d}F_{\boldsymbol{U}}(\boldsymbol{u})\right)^{-1}.$$

Write $\boldsymbol{u}\in\mathcal{S}_2$ in polar coordinates as $\boldsymbol{u}=(\cos\theta,\sin\theta)$, $\theta\in[-\pi,\pi]$. As in the proof of Theorem 2(ii) we obtain

$$\lim_{C\to-\infty}\mathbf{P}(Z_1\le D_1|Z_0\le C)=\int_{-\arcsin\rho_{01}}^{\pi/2}(\cos\theta)^{\alpha}\,\mathrm{d}\theta\left(\int_{-\pi/2}^{\pi/2}(\cos\theta)^{\alpha}\,\mathrm{d}\theta\right)^{-1}.$$

Using the definitions of the incomplete beta function $B(z;a,b)$ and the regularised incomplete beta function $I_x(a,b)$

$$\begin{aligned}B(z;a,b)&:=\int_0^z u^{a-1}(1-u)^{b-1}\,\mathrm{d}u=2\int_0^{\arcsin(\sqrt{z})}(\sin\theta)^{2a-1}(\cos\theta)^{2b-1}\,\mathrm{d}t\\ I_x(a,b)&:=\frac{B(x;a,b)}{B(a,b)},\end{aligned}$$

where $B(a,b):=B(1,a,b)$, yields

$$\begin{aligned}\int_{-\pi/2}^{\pi/2}(\cos\theta)^{\alpha}\,\mathrm{d}\theta&=B\left(\frac{1}{2},\frac{\alpha+1}{2}\right),\\ \int_{-\arcsin\rho_{01}}^{\pi/2}(\cos\theta)^{\alpha}\,\mathrm{d}\theta&=\int_0^{\arcsin\rho_{01}}(\cos\theta)^{\alpha}\,\mathrm{d}\theta+\frac{1}{2}B\left(\frac{1}{2},\frac{\alpha+1}{2}\right)\\ &=\frac{1}{2}B\left(\rho_{01}^2;\frac{1}{2},\frac{\alpha+1}{2}\right)+\frac{1}{2}B\left(\frac{1}{2},\frac{\alpha+1}{2}\right)\end{aligned}$$

and therefore

$$\lim_{C\to-\infty}\mathbf{P}(Z_1\le D_1|Z_0\le C)=\frac{1}{2}+\frac{1}{2}I_{\rho_{01}^2}\left(\frac{1}{2},\frac{\alpha+1}{2}\right).$$

The claim follows since this expression corresponds to the respective Student t-distribution function via the following relationship between an incomplete beta function and the distribution function t_ν of the Student-t distribution with parameter ν:

$$t_\nu(x)=\begin{cases}\frac{1}{2}I_{\nu/(x^2+\nu)}\left(\frac{\nu}{2},\frac{1}{2}\right), & x\le 0,\\ \frac{1}{2}\left[1+I_{x^2/(x^2+\nu)}\left(\frac{1}{2},\frac{\nu}{2}\right)\right], & x>0.\end{cases}$$

For *(ii)*, the proof follows in a similar way, but is computationally more involved: write $\boldsymbol{u}\in\mathcal{S}_3$ in polar coordinates, insert this into the integral representation and re-write in terms of (incomplete) beta functions, resp. Student-t distributions. □

3.3 Default correlations under stress

In the case where $\mathbf{P}(G>\cdot)\in RV_{-\alpha}$, default correlations can be explicitly calculated from Theorem 4. For the case where G is rapidly varying, we have the following result.

Theorem 5. *Let $\mathbf{P}(G>\cdot)\in RV_{-\infty}$. Then,*

$$\lim_{C\to-\infty}Corr^C(\mathbf{1}_{\{Z_1\le D_1\}},\mathbf{1}_{\{Z_2\le D_2\}})=0,$$

where $Corr^C$ denotes the correlation under $\mathbf{P}(\cdot|Z_0\le C)$.

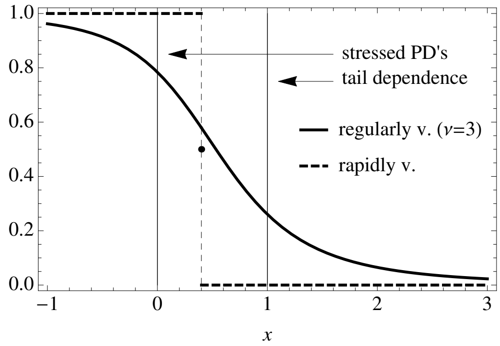

Figure 1: Tail dependence function $\lambda(Z_0, Z_1, x)$ for light- and heavy-tailed variables; special cases arise at $x = 0$ (stressed PD's) at $x = 1$ (tail dependence). The correlation parameter is 0.4 in both cases.

The proof follows along the lines of the proof of Lemma 3 and Theorem 2. Interestingly, this limit behaviour arises for light-tailed asset returns Z_1, Z_2 even when the asset correlation $\mathrm{Corr}(Z_1, Z_2)$ is strictly smaller than 1, but converges to 1 conditional on $Z_0 \leq C$ and as $C \to -\infty$; see Kalkbrener and Packham (2015a) for analytic formulas for the asymptotic asset correlation.

3.4 Default probabilities and tail dependence

Tail dependence is a popular measure in finance to assess the ability of a bivariate distribution to generate joint extreme events. For two random variables Y_1 and Y_2 with distribution functions F_1 and F_2, the *coefficient of (lower) tail dependence* of Y_1 and Y_2 is defined as

$$\lambda_l(Y_1, Y_2) := \lim_{q \to 0^+} \mathbf{P}(Y_2 \leq F_2^{\leftarrow}(q) | Y_1 \leq F_1^{\leftarrow}(q)), \tag{6}$$

where $F_i^{\leftarrow}$ denotes the inverse of the df F_i.

A closed-form expression for the tail dependence coefficient of elliptically distributed random variables can be found e.g. in Hult and Lindskog (2002); Schmidt (2002); McNeil *et al.* (2005). The tail dependence is positive for heavy-tailed and zero for light-tailed elliptical distributions; the latter includes, of course, the normal distribution. In structural credit portfolio models, such as CreditMetrics$^{\mathrm{TM}}$, Gupton *et al.* (1997), and Moody's KMV Portfolio Manager$^{\mathrm{TM}}$, Crosbie and Bohn (2002), the multivariate normal distribution is still the de-facto standard for modelling risk factors and asset log-returns. The zero tail dependence is in contrast to the asymptotic default probability in the light-tailed case, where default is a sure event. Because of the zero tail dependence, normally distributed models are considered less sensitive to stress than heavy-tailed models. The preceding results show that this need not be the case: in the limit, the impact of stress on default probabilities is greater in light-tailed models than in heavy-tailed models.

To make the relation between tail dependence and asymptotic stressed PD's more precise, we consider the following function

$$\lambda(Y_1, Y_2, x) := \lim_{C \to -\infty} \mathbf{P}(Y_2 \leq x\, C | Y_1 \leq C), \quad x \in \mathbb{R},$$

which provides an elegant generalization of both concepts. Since Z_0 and Z_1 are identically distributed, their tail dependence coefficient equals $\lambda(Z_0, Z_1, 1)$, whereas the asymptotic stressed PD's correspond to $\lambda(Z_0, Z_1, 0)$.

Using elementary transformations we obtain the following representations from Theorem 1 in Abdous *et al.* (2005): for an elliptically distributed random vector (Z_0, Z_1) with mixing variable G in $RV_{-\alpha}$,

$$\lambda(Z_0, Z_1, x) = \begin{cases} t_{\alpha+1}\left(\frac{\sqrt{\alpha+1}\,(\rho-x)}{\sqrt{1-\rho^2}}\right) + \frac{\operatorname{sgn}(x)}{|x|^\alpha}\, t_{\alpha+1}\left(\frac{\operatorname{sgn}(x)\sqrt{\alpha+1}\,(\rho-1/x)}{\sqrt{1-\rho^2}}\right), & \text{if } x \neq 0, \\ t_{\alpha+1}\left(\frac{\sqrt{\alpha+1}\,\rho}{\sqrt{1-\rho^2}}\right), & \text{if } x = 0 \end{cases}$$

is a continuous function, whereas for G in the Gumbel max-domain of attraction,

$$\lambda(Z_0, Z_1, x) = \begin{cases} 1, & \text{if } x < \rho, \\ 1/2, & \text{if } x = \rho, \\ 0, & \text{if } x > \rho. \end{cases}$$

This is illustrated in Figure 1.

The analysis of the function $\lambda(Z_0, Z_1, x)$ illustrates the fundamentally different behaviour of the tail dependence coefficient and asymptotic stressed PD's in light-tailed and heavy-tailed credit portfolio models. In the rapidly varying case, the random variable Z_1 converges to $-\infty$, more specifically, it is concentrated at ρC when $Z_0 \leq C$ and $C \to -\infty$, i.e.,

$$\lim_{C \to -\infty} \mathbf{P}((\rho+\varepsilon)\, C \leq Z_1 \leq (\rho-\varepsilon)\, C | Z_0 \leq C) = \lambda(Z_0, Z_1, \rho-\varepsilon) - \lambda(Z_0, Z_1, \rho+\varepsilon) = 1,$$

for arbitrary $\varepsilon > 0$. As a consequence, the tail dependence of Z_0 and Z_1 is 0, whereas stressed PD's converge to 1. In the regularly varying case, however, Z_1 does not show the same uniform asymptotic behaviour: $0 < \lambda(Z_0, Z_1, x) < 1$ holds for all $x \in \mathbb{R}$ and, in particular, tail dependence as well as stressed default probabilities are in $(0, 1)$.

By contrast, for stressed default correlations, we observe a behaviour similar to tail dependence: we showed earlier that stressed default correlations converge to a positive number in the heavy-tailed case and to 0 in the light-tailed case.

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